Universal twinkling exponents for spectral fluctuations associated with mixed chaology

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For systems that are neither fully integrable nor fully chaotic, bifurcations of periodic orbits give rise to semiclassically emergent singularities in the fluctuating part $\mathcal{N}_{\rm fl}$ of the energy-level counting function. The bifurcations dominate the spectral moments $M_m(\hbar) = \left\langle (\mathcal{N}_{\rm fl})^{2m} \right\rangle$ in the limit $\hbar \to 0$. We show that $M_m(\hbar) \sim \text{constant}/\hbar^{\nu_m}$, and calculate the twinkling exponents ν_m as the result of a competition between bifurcations with different codimensions and repetition numbers.

I. INTRODUCTION

Our aim here is to characterize energy-level fluctuations in quantum systems whose classical counterparts are mixed, that is, neither completely integrable nor completely chaotic. Previous work [1] indicates that the short-range fluctuations can be usefully approximated by a superposition of the Poisson and random-matrix spectral statistics that respectively describe integrable and chaotic systems [2–5]. Here we will argue that there is a complementary description, more fundamentally associated with the mixed regime.

The new description is associated with bifurcations, where combinations of stable and unstable orbits collide and transform into others, or annihilate, as a parameter (for example energy) varies—it is the ubiquity of bifurcations, after all, that characterizes mixed systems. The main result will be the prediction that the spectral moments—describing the fluctuations in the distribution of energy levels as explained below—are dominated by a competition among the different sorts of bifurcation.

Bifurcations are singularities of the dynamics, and the statistics to be calculated here are a new example of the wider class of 'singularity-dominated strong fluctuations'. This old but still-unfamiliar idea is that some variables exhibit wild (non-gaussian) fluctuations, with very large values described by scaling laws and associated with particular geometric singularities. For a review, see [6] (but note that some of the exponents in section 4 of this publication are wrong, and superseded by the present paper). The fluctuations most closely analogous to those we consider here are the intensity variations of twinkling starlight, where the short-wave singularities are caustics, and the intensity moments depend on a competition [7] among catastrophes (universality classes of caustic). Although the formal analogy between spectral fluctuations and light caustics is close, orbit bifurcations are classified differently, and in their technical aspects, and their results, the two theories diverge.

Like all statistics related to chaology, those described here emerge semiclassically, that is in the limit of vanishing Planck's constant $\hbar$. Consider a set of levels $\{E_1(\hbar), E_2(\hbar), \ldots, E_j(\hbar), \ldots\}$. This spectrum can be characterized by the counting function, or spectral staircase:

$$\mathcal{N}(E,\hbar) = \sum_{j=1}^{\infty} \Theta(E - E_j(\hbar)) \ , \tag{1}$$

where Θ denotes the unit step. As usual, we separate $\mathcal{N}$ into its smooth and fluctuating parts:

$$\mathcal{N}(E,\hbar) = \mathcal{N}_{\rm sm}(E,\hbar) + \mathcal{N}_{\rm fl}(E,\hbar) \ . \tag{2}$$

$\mathcal{N}_{\rm sm}$ is given by the Weyl rule plus $\hbar$-corrections [8].

We will concentrate on the semiclassical size of the spectral fluctuations $\mathcal{N}_{\rm fl}$, as embodied in the spectral moments

$$M_m(\hbar) = \left\langle [\mathcal{N}_{\rm fl}(E,\hbar)]^{2m} \right\rangle \ . \tag{3}$$

Here $\langle \cdots \rangle$ denotes a local energy average for an individual hamiltonian. However, central to our calculation will be the replacement of the energy average by averages over parameters for families of hamiltonians including the given one. This implied ergodicity is implicit in many semiclassical arguments (for example, it leads directly to the short-range level repulsion for different classes of systems [2]). The main result will be that

$$M_m(\hbar) \sim \frac{\text{constant}}{\hbar^{\nu_m}} \ \text{(up to logarithms) as } \hbar \to 0 \ , \tag{4}$$

where ν_m are the 'twinkling exponents': universal numbers that we will determine by studying the hierarchy of bifurcations. Each exponent can be determined as the slope on a log-log plot, that is

$$\lim_{\hbar \to 0} \frac{\partial \log\{M_m(\hbar)\}}{\partial \log\{1/\hbar\}} = \nu_m \ . \tag{5}$$

Our calculation will be for systems with two freedoms. For these, a strict upper bound $\nu_m \leq 4m$ follows from the Weyl rule $\mathcal{N} \sim \mathcal{N}_{\rm sm} \sim \text{constant}/\hbar^2$, implying

$$\hbar^2 \mathcal{N}_{\rm fl} \to 0 \text{ as } \hbar \to 0 \ . \tag{6}$$

II. SEMICLASSICAL THEORY

To see the importance of bifurcations, we first recall the trace formulas, in which $\mathcal{N}_{\mathrm{fl}}(E)$ can be represented semiclassically as a sum over periodic orbits. In the generic case, where orbits are isolated, the sum is over primitive periodic orbits p with energy E, and their repetitions r [9,10]:

$$\mathcal{N}_{\mathrm{fl}}(E,\hbar) \approx \frac{1}{\pi} \sum_p \sum_{r=1}^{\infty} \frac{\sin\{S_{p,r}(E)/\hbar - \mu_{p,r}\}}{r\sqrt{|\det([\mathbf{M}_p(E)]^r - 1)|}} \ . \qquad (7)$$

Here $S_{p,r}$ is the action of the orbit, $\mathbf{M}_p$ is the monodromy matrix describing the linearised return map on the Poincaré section, and $\mu_{p,r}$ is the Maslov index (which will play no further part in our reasoning).

In the integrable case, for two freedoms, with hamiltonian $H(\mathbf{I})$ involving action variables $\mathbf{I} = \{I_1, I_2\}$, the sum is over resonant tori characterized by their winding numbers $\mathbf{W} = \{W_1, W_2\}$, and the trace formula is [11]

$$\mathcal{N}_{\mathrm{fl}}(E,\hbar) \approx \frac{1}{\pi\hbar^{1/2}} \sum_{\mathbf{W}} \frac{\sin\{2\pi\mathbf{W}\cdot\mathbf{I}_{\mathbf{W}}(E)/\hbar - \mu_{\mathbf{W}}\}}{|\mathbf{W}|^{3/2}\sqrt{K(\mathbf{I}_{\mathbf{W}}(E))}} \ . \quad (8)$$

Here $\mathbf{I}_{\mathbf{W}}(E)$ are the actions of the resonant tori, where the frequencies ω are commensurate, and K is the curvature of the energy contour $H(\mathbf{I}) = E$ in $\mathbf{I}$ space.

When these formulas apply, there are no strong fluctuations and the estimation of $\mathcal{N}_{\mathrm{fl}}$ is fairly simple. For the chaotic case (7), the prefactor is of order $\hbar^0$; the sum diverges, but can be regularized by truncation at orbits with period equal to the Heisenberg time $\hbar/$(mean level spacing), which, together with the exponential proliferation of orbits with increasing period, leads to

$$|\mathcal{N}_{\mathrm{fl}}| \sim \text{constant} \times \sqrt{\log(1/\hbar)} \ , \qquad (9)$$

and moments (5) with all twinkling exponents $\nu_m = 0$. For the two-dimensional integrable case (8), where the sum converges,

$$|\mathcal{N}_{\mathrm{fl}}| \sim \text{constant} \times \hbar^{-1/2} \ , \qquad (10)$$

and (3) and (5) give $\nu_m = m$.

Here we are interested in cases when the trace formulas fail. This happens at bifurcations of periodic orbits. In (7), bifurcations of isolated orbits correspond to a unit eigenvalue of the monodromy matrix, so that $\det(\mathbf{M} - 1)$ vanishes, and the terms representing those orbits diverge. In (8) bifurcations of tori correspond to coalescence of parallel normals to the energy surface, so that K vanishes and the terms representing those orbits diverge. The formulas fail, but it is clear that bifurcations lead to large values of $\mathcal{N}_{\mathrm{fl}}$. How large? This has been studied by several authors [12–18], who have found corrected versions of the trace formula that incorporate the bifurcations properly, with the result that $\mathcal{N}_{\mathrm{fl}}$ does not diverge but rises to values that increase as $\hbar \to 0$. We will extend these results to estimate the moments $M_m(\hbar)$ and hence the twinkling exponents ν_m.

Near bifurcations, the trace formulas (7) and (8) must be replaced by the 'diffraction integrals' for which they are the stationary-phase approximations. Before writing these, we note that, for two freedoms, periodic orbits are fixed points of the map determined by successive intersections with the Poincaré section with coordinates q, p. In terms of the generating function ϕ, the map can be specified as

$$\begin{pmatrix} q \\ p \end{pmatrix} \to \begin{pmatrix} q' \\ p' \end{pmatrix} \ , \ \text{with } q = \partial_p \phi(q',p) \ , \ p' = \partial_{q'}\phi(q',p) \ . \qquad (11)$$

Thus, periodic orbits are critical points of the reduced generating function (henceforth called generator)

$$\begin{aligned} &\Phi(q',p) = \phi(q',p) - q'p \ , \\ &\{\partial_q \Phi(q,p) = \partial_p \Phi(q,p) = 0\} \leftrightarrow \{q' = q \ , \ p' = p\} \ . \end{aligned} \qquad (12)$$

To write the diffraction integral describing semiclassical spectral fluctuations, we need the generator Φ_r of the r-times-iterated Poincaré map. ($\Phi_r(q,p;E)$ can be regarded as an effective hamiltonian describing the motion between r intersections of orbits with the Poincaré section.)

Up to irrelevant factors, the fluctuations are [12,15]

$$\begin{aligned} &\mathcal{N}_{\mathrm{fl}}(E) \approx \sum_{r=1}^{\infty} \mathcal{N}_{\mathrm{fl},r}(E) \ , \\ &\mathcal{N}_{\mathrm{fl},r}(E) = \mathrm{Im}\,\frac{1}{\hbar}\int\!\!\int \mathrm{d}q\,\mathrm{d}p\ \exp\left\{\frac{i}{\hbar}\Phi_r(q,p;E)\right\} \ . \end{aligned} \qquad (13)$$

Provided the sum over r converges, this semiclassical theory implies the bound $\nu_m \le 2m$, sharper than the strict bound $\nu_m \le 4m$ obtained at the end of section I.

Periodic orbits correspond to stationary values of the phases in these integrals. If the stationary points are isolated, the stationary-phase approximation reproduces (7). If the system is integrable, Φ depends not on q and p separately but on a combination (action variable) such as $q^2 + p^2$, and is stationary on lines, corresponding to resonant tori; then stationary-phase reproduces (8). At bifurcations, where isolated periodic orbits or resonant tori coalesce, these approximations fail. For different sorts of bifurcation, the patterns of coalescence are different, and $\mathcal{N}_{\mathrm{fl}}$ can be described locally [12] by replacing Φ by an appropriate normal form (the validity of the description can be extended by approximating the integrals (13) by the technique of uniform approximations [15,16,18], but that is not required for our purposes).

III. NORMAL FORMS AND SCALING

We envisage that for each bifurcation the local generator for each repetition number r depends on parameters

$\mathbf{x} = \{x_n\}$ $(1 \le n \le K)$ in addition to q and p; one of these parameters is the energy E, and K is the codimension of the singularity. The parameters describe the unfolding of the bifurcation, that is, the ways in which the degenerate periodic orbits can split into combinations of nondegenerate orbits. Reflecting this, we denote the normal forms for the bifurcations by

$$\Phi_r = \Phi_{r,K}(q, p; \mathbf{x}) \ , \tag{14}$$

and define the associated canonical integrals

$$\mathcal{N}_{\mathrm{fl},r,K}(\mathbf{x}, \hbar) \equiv \mathrm{Im}\, \frac{1}{\hbar} \int\!\!\int \mathrm{d}q\, \mathrm{d}p\, \exp\left\{\frac{i}{\hbar}\Phi_{r,K}(q,p;\mathbf{x})\right\} \ . \tag{15}$$

The strategy now is to simplify the $\hbar$ dependence of these integrals in a way that enables the averages in the moments (3) to be estimated as integrals over the parameters $\mathbf{x}$. This will be achieved by a two-stage process: rescaling the integration variables q and p to remove the $1/\hbar$ factor from the dominant term (germ) of the generator in the exponent of (15), and then applying compensating rescaling of the parameters $\mathbf{x}$. This will lead to

$$\mathcal{N}_{\mathrm{fl},r,K}(\mathbf{x}, \hbar) = \frac{1}{\hbar^{\beta_{r,K}}} \mathcal{N}_{\mathrm{fl},r,K}\left(\{x_n/\hbar^{\sigma_{n,r,K}}\}, 1\right) \ . \tag{16}$$

The exponent β describes the semiclassical strength of the spectral fluctuations at the bifurcation. The exponents σ describe the scale of the interference fringes associated with nondegenerate periodic orbits that appear in the different unfolding directions x_n. We will also need the associated exponent

$$\gamma_{r,K} = \sum_{n=1}^{K} \sigma_{n,r,K} \ , \tag{17}$$

describing the scaling of the K-dimensional $\mathbf{x}$ space hypervolume associated with interference near the bifurcation.

Armed with the scaling law (16), we can estimate the contribution of the bifurcation r, K to the ensemble average for the mth moment (3). This is

$$\begin{aligned} M_{m,r,K} &\equiv B \int \mathrm{d}^K\mathbf{x}\, \mathcal{N}_{\mathrm{fl},r,K}(\mathbf{x}, \hbar)^{2m} \\ &= \frac{B}{\hbar^{(2m\beta_{r,K} - \gamma_{r,K})}} \int \mathrm{d}^K\mathbf{y}\, [\mathcal{N}_{\mathrm{fl},r,K}(\mathbf{y}, 1)]^{2m} \ , \end{aligned} \tag{18}$$

where B is a normalization constant.

With the $\hbar$ dependence thus extracted, these contributions can now be compared for the different bifurcations. The dominant contribution(s) will come from the bifurcation(s) with the strongest $\hbar$ dependence, leading to the fluctuation moment scaling law (4) and (5) as the result of a competition among bifurcations, resulting in the twinkling exponents

$$\nu_m = \max_{(r,K)}(2m\beta_{r,K} - \gamma_{r,K}) \ . \tag{19}$$

Here we will consider only the fully generic situation where the dynamics is such that all bifurcations occur in the neighbourhood of the system under consideration. Then the competition in (19) is unrestricted. If for some reason (e.g. symmetry) some classes of bifurcation are forbidden, the competition must be appropriately restricted, and the resulting exponents will be different. Analogous restricted competitions have been explored in the optical context [19] in the analysis of an experiment to measure twinkling exponents.

To carry out this program, we need the normal forms of the bifurcations labelled r, K. For $r = 1$, these are the elementary catastrophe polynomials representing the different ways that critical points of smooth functions can coalesce [20–23], and the exponents in the scaling law (16) and (17) have already been calculated [7]. This is analogous to the optical case, where the appropriate diffraction integral is the first term $r = 1$ of the sum (13). For the cuspoid catastrophes, where one variable (p, say) is quadratic—in the language of catastrophe theory, these are catastrophes of corank 1—the normal forms are

$$\Phi_{1,K} = p^2 + q^{K+2} + \sum_{n=1}^{K} x_n q^n \tag{20}$$

(any term of order q^{K+1} can be eliminated by shifting the origin) and the exponents are

$$\begin{aligned} \beta_{1,K}^{\mathrm{cuspoids}} &= \frac{K}{2(K+2)} \ , \ \sigma_{1,K}^{\mathrm{cuspoids}} = 1 - \frac{n}{K+2} \ , \\ \gamma_{1,K}^{\mathrm{cuspoids}} &= \frac{K(K+3)}{2(K+2)} \ . \end{aligned} \tag{21}$$

We do not give the more complicated expressions corresponding to catastrophes of corank 2, where the generators involve both q and p nontrivially.

When $r \ge 1$, however, the normal forms are not the elementary catastrophes, because the period-r generator must have the special property of possessing an rth root, namely the generator for the primitive map. Some information is available for bifurcations of period-r orbits with $K = 1$ [24–27] and $K = 2$ [17], but this is not sufficient for our purposes.

To get the results we need, we start by transforming to polar coordinates in phase space, that is

$$q = \sqrt{I/2}\cos\phi \ , \ p = \sqrt{I/2}\sin\phi \ , \tag{22}$$

and noting that the generators for period-r bifurcations must have ϕ dependence with period $2\pi/r$, so the ϕ-dependent terms of lowest degree in I must involve $\cos(r\phi)$ and $\sin(r\phi)$. Moreover, the generators must be smooth functions of q and p at the origin, which excludes terms $I^s\cos(r\phi)$ with $s < r/2$. This leads to the surprising conclusion that if $r \ge 2K + 2$ the ϕ-dependent terms are all of higher order than the unfolding terms containing the parameters $\mathbf{x}$. Thus we can write

$$\Phi_{r,K}(q,p;\mathbf{x}) = I^{K+1} + \sum_{n=1}^{K} x_n I^n \quad (r \geq 2K+2) \ . \tag{23}$$

(The resemblance to the cuspoid generators (20) is misleading: it is not legitimate to eliminate the highest unfolding term $x_k I^K$ by shifting the origin of I, since this would violate the condition that I must be nonnegative—alternatively stated, the origin of I is privileged, unlike the origin of q in (20).)

Reverting to q and p, and scaling the integrals (15), we get, for the exponents in (16) and (17),

$$\beta_{r,K} = \frac{K}{K+1} \ , \ \sigma_{n,r,K} = 1 - \frac{n}{K+1} \ ,$$
$$\gamma_{r,K} = \frac{1}{2}K \qquad (r \geq 2K+2) \ . \tag{24}$$

Note that these exponents do not involve r.

In (23) we are neglecting the ϕ-dependent terms, but we are not asserting their absence—of course these terms must be present, to describe the 'island necklaces' of stable and unstable orbits into which the degenerate orbits bifurcate. But because the neglected terms are of higher order (reflecting the fact that the islands are very thin close to the bifurcation) the parameters that would multiply them acquire negative exponents σ under scaling, and so disappear semiclassically from the diffraction integrals. Alternatively stated in the language of critical phenomena, these parameters are irrelevant variables. As a simple illustrative example, consider $K = 1$, $r = 5$ (i.e. $r \geq 2K+2$). Then the generator, including the leading ϕ-dependent term, is

$$\Phi_{5,1}(q,p;\mathbf{x}) = I^2 + x_1 I + x_5 I^{5/2}\cos(5\phi) \tag{25}$$
$$= 4(p^2+q^2)^2 + 2x_1(p^2+q^2) + 2^{5/2}x_5 \,\mathrm{Re}\,(q+ip)^5 \ .$$

Scaling $\hbar$ from the exponent in (15) gives $\beta = 1/2$, and incorporating this into the parameter x_1 gives $\sigma_1 = 1/2$. However, applying the same scaling to the 'necklace' parameter x_5 gives $\sigma_5 = -1/4$, which is negative and therefore irrelevant. (If $r = 2K+2$, the leading necklace parameter gives the marginal exponent $\sigma = 0$, which does not affect any of our subsequent arguments.)

We have not determined the generators for $1 < r < 2K+2$, but will soon argue that these bifurcations cannot contribute to the twinkling exponents.

m	2	3	4	5	6	7	8	9	10	11	12
ν_m	$\frac{5}{3}$	3	$\frac{9}{2}$	6	$\frac{38}{5}$	$\frac{46}{5}$	$\frac{65}{6}$	$\frac{25}{2}$	$\frac{85}{6}$	$\frac{111}{7}$	$\frac{123}{7}$
dominating K	2	2,3	3	3,4	4	4	5	5	5	6	6

TABLE I. Twinkling exponents ν_m, and the codimension(s) K of the dominating bifurcations(s), for generic two-freedom systems.

IV. BATTLE OF BIFURCATIONS

Suppose for the moment that all relevant bifurcations have $r \geq 2K+2$, so that (23) applies. Then the twinkling exponents are determined by the competition (19), where the entrants are the β and γ values in (24), that is

$$\nu_m = \max_K \left(\frac{2mK}{K+1} - \frac{1}{2}K \right) \ . \tag{26}$$

The results are given in table I.

Now we will argue that these results are unaffected by allowing the bifurcations with $r < 2K+1$ to enter the competition. This requires the twinkling exponents associated with this class of singularities to be smaller than those in table I. The generators for $r < 2K+2$ will contain ϕ-dependent unfolding terms, and can be written in the form

$$\Phi_{r,K}(q,p;\mathbf{x}) = I^{l(K)+1} + \sum_{n=1}^{l(K)} x_n I^n$$
$$+ \sum_{n=l(K)+1}^{K} (\text{terms involving } \phi) \ , \tag{27}$$

where $l(K) < K$ and all the terms involving ϕ are of lower degree than $I^{l(K)+1}$. For each such generator, there is a partner in the class with $r \geq 2K+2$, of the form (23), with $K' = l(K)$. This partner has the term germ $I^{l(K)+1}$ as (27), and therefore the same exponent β, but its γ exponent is smaller, because of the additional terms in (27). Therefore the partner with $r \geq 2K+2$ has the larger twinkling exponent $2m\beta - \gamma$, and so dominates the competition.

This general argument can be verified directly for the special case of bifurcations with $r = 1$, namely the elementary catastrophes. For the cuspoids (corank 1), with exponents (21), it is easy to calculate the results of the competition (19); this has already been done in the optical context [7], and all exponents are indeed smaller than those in table I. The same is true for the corank 2 catastrophes, even though the exponents are all larger than for corank 1 (for corank 2 the classification is incomplete, but the conclusion holds for all classes of singularities that have been examined).

Thus the entries in table I are confirmed as the universal twinkling exponents associated with bifurcations of generic systems with 2 freedoms.

V. DISCUSSION

We have argued that to leading order in $1/\hbar$, the spectral fluctuation moments M_m diverge according to power-laws, with exponents—those winning the competition (19)—given in table I. This goes far beyond the already established fact that bifurcations contribute to the spectral statistics when regular and chaotic orbits coexist [28], because these new semiclassical fluctuation phenomena involve many competing bifurcations, not just one.

Two observations may assist the eventual observation of the universal fluctuations we are predicting. The first relates to our concentration on the average effect of large spectral fluctuations associated with individual bifurcations of periodic orbits with finite length, while ignoring possible collective effects of long orbits. But this collective effect seems small (equation 9 and the remark following it), so we expect the associated fluctuations to contribute only a weak background that will not mask the bifurcation fluctuations we are interested in.

The second observation is that although all the twinkling exponents in table I satisfy the inequality $\nu_m \leq 2m$, all the exponents with $m > 3$ exceed the value $\nu_m = m$ for integrable systems, rendering unnecessary the problematic subtraction of possible contributions to the fluctuations from the (nonresonant) KAM tori that persist in the systems we have been studying here.

Nevertheless, it is difficult to make quantitative predictions of the circumstances in which the twinkling exponents might be seen in computer or laboratory experiments. To illustrate this, suppose that the dominating bifurcation, with exponent ν_m, has an associated coefficient A_m, and the runner-up in the competition has exponent $\nu_{1m} < \nu_m$ and coefficient A_{1m}. Then the two leading terms in the moment asymptotics will be

$$M_m(\hbar) \sim \frac{A_m}{\hbar^{\nu_m}} + \frac{A_{1m}}{\hbar^{\nu_{1m}}} \,. \tag{28}$$

If it should happen that $A_m \ll A_{1m}$, and ν_m exceeds ν_{1m} only slightly, experiments will indicate the wrong exponents ν_{1m} unless $\hbar$ is less than the crossover value

$$\hbar = \left(\frac{A_m}{A_{1m}}\right)^{1/(\nu_m - \nu_{1m})} \,, \tag{29}$$

which in the circumstances indicated is very small.

Our reasoning hints at fabulous complexity in the full semiclassical asymptotics of the moments: many different bifurcations contribute according to (18), and these terms are merely the leading orders in (almost certainly divergent) $\hbar$ expansions, because the normal forms of the generators give only local approximations to the diffractions integrals (13), which themselves are lowest-order semiclassical approximations. These observations lead to the expectation that

$$M_m(\hbar) = \sum_{r,K} \frac{A_{m,r,k}}{\hbar^{(2m\beta_{r,k} - \gamma_{r,k})}} \left(1 + \sum_{s=1}^{\infty} \alpha_{s,m,r,K}\hbar^s\right) \tag{30}$$

(where the A and α coefficients might involve logarithms of $1/\hbar$).

We know nothing about the leading-order bifurcation coefficients $A_{m,r,K}$ or the corrections $\alpha_{s,r,m,K}$. These coefficients are not universal, so calculating them would require detailed knowledge of the individual bifurcations in the particular dynamical system being considered. Even for the less complicated case of optical twinkling, the A coefficients have been calculated only for the simplest situation: cuspoids in diffraction from a corrugated phase-changing screen [29,30].

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