Quantum Information I: entanglement

- EPR paradox & entanglement measures B:20.1—KP:18.1,19.3.1
- Bell inequality B:20.2,20.3—KP:19.4.1
- teleportation

B = Ballantine — KP = Konishi & Paffuti

EPR paradox & entangled states

Einstein, Podolsky & Rosen (1935): "spooky action at a distance"

$$\Psi = \frac{1}{\sqrt{2}}|\uparrow\rangle_A|\downarrow\rangle_B - \frac{1}{\sqrt{2}}|\downarrow\rangle_A|\uparrow\rangle_B$$

when B measures $\downarrow$ the state of A collapses "instantaneously" to $\uparrow$

actually, there is no violation of the principle of relativity, as follows by comparing the reduced density matrix $\rho_A = \mathrm{Tr}_B\,\rho$ before and after the measurement: in both cases the answer is the same,

$$\rho_A = \tfrac{1}{2}|\uparrow\rangle\langle\uparrow| + \tfrac{1}{2}|\downarrow\rangle\langle\downarrow|$$

$\rho_A^2 \neq \rho_A$, so A is in a mixed state, while A en B together were in a pure state: A and B are entangled

product state $|\uparrow\rangle_A|\downarrow\rangle_B \Rightarrow \rho_A = |\uparrow\rangle\langle\uparrow|$ remains pure.

entanglement measures

must be invariant under local unitary transformations

two-level system *in a pure state*:

$$\Psi = \alpha|\uparrow\rangle_A|\downarrow\rangle_B + \beta|\downarrow\rangle_A|\uparrow\rangle_B, \quad |\alpha|^2 + |\beta|^2 = 1.$$

the product $C = 2|\alpha\beta|$ (the concurrence) measures how far Ψ is from a product state: $C = 0$ for a product state and $C = 1$ for a maximally entangled state.

more generally, for $\Psi = \sum_{nm} c_{nm}|n\rangle_A|m\rangle_B$ one has $C = 2|\det c|$ — see exercise 1

2^N levels: entanglement entropy

$$S = -\mathrm{Tr}\,\rho_A{}^2 \log \rho_A = -\mathrm{Tr}\,\rho_B{}^2 \log \rho_B$$

varies between 0 (ρ_A is pure) and N ($\rho_A = 2^{-N} \times$ unit matrix)

$\rho_A \mapsto cc^\dagger$ and $\rho_B \mapsto c^T c^*$ have the same eigenvalues ρ_n and thus the same $S = -\sum_n \rho_n \log \rho_n$

Bell inequality

the spins of A and B in the entangled state
$\Psi = \frac{1}{\sqrt{2}}|\uparrow\rangle_A|\downarrow\rangle_B - \frac{1}{\sqrt{2}}|\downarrow\rangle_A|\uparrow\rangle_B$ are *anticorrelated in any basis,*

$$\langle \hat{a}\hat{b}\rangle \equiv \langle\Psi|(\hat{a}\cdot\sigma_A)(\hat{b}\cdot\sigma_B)|\Psi\rangle = -\hat{a}\cdot\hat{b} = -\cos\theta$$

John Bell (1964): method to distinguish classical from quantum correlations, by comparing correlations in different basis.

take four coplanar unit vectors $\hat{a}'$, $\hat{b}$, $\hat{a}$, $\hat{b}'$ separated by 45°:

$\langle \hat{a}\hat{b}\rangle = \langle \hat{a}'\hat{b}\rangle = \langle \hat{a}\hat{b}'\rangle = -\cos\pi/4 = -1/\sqrt{2}$, while $\langle \hat{a}'\hat{b}'\rangle = -\cos 3\pi/4 = 1/\sqrt{2}$, so

$\mathcal{B} \equiv \langle \hat{a}\hat{b}\rangle + \langle \hat{a}'\hat{b}\rangle + \langle \hat{a}\hat{b}'\rangle - \langle \hat{a}'\hat{b}'\rangle = -2\sqrt{2}$.

But classically, $|\mathcal{B}| \leqslant 2$, because if $A, B, A', B' \in \{\pm 1\}$, then $AB + A'B + AB' - A'B' = A(B+B') + A'(B-B') = \pm 2$, so the average is between -2 and $+2$.

this Bell inequality is tested experimentally with photon polarizations, eliminating "hidden variable theories".

teleportation

see exercise 2