Critical behaviour of Ising systems on extremely anisotropic lattices

C. A. W. Citteur

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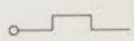
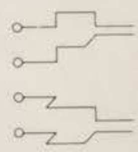
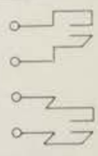
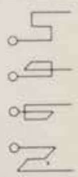
+13		1		1 1 1 1
+14		1/2		1/2 1 1 1/2
+21		1		1 1 1 1

CRITICAL BEHAVIOUR OF ISING SYSTEMS ON EXTREMELY ANISOTROPIC LATTICES

Proefschrift

ter verkrijging van de graad van doctor in

de wiskunde en natuurwetenschappen aan de

Rijksuniversiteit te Leiden, op gezag van

de Rector Magnificus Dr. A.E. Cohen ,

hoogleraar in de faculteit der letteren,

volgens besluit van het college van dekanen

te verdedigen op woensdag 11 april 1973

te klokke 16.15 uur

door

Cornelis Arthur William Citteur

geboren te 's-Gravenhage in 1940

1973

Beugelsdijk - Leiden

Promotor: Prof.Dr. P.W. Kasteleyn

1. Het verdient aanbeveling aan stoffen waarvan bekend is dat zij in zeer goede benadering als Ising-systemen kunnen worden beschouwd experimenten te verrichten waarbij het kritische gedrag wordt bestudeerd in gevallen waarin de stof gelijktijdig aan temperatuurveranderingen en aan een grote uitrekking is onderworpen.

2. De initiële susceptibiliteit van een Ising-systeem op een Cartesisch rooster waarin de koppelingsconstante voor naaste buren langs één roosteras negatief en in absolute waarde veel groter is dan de andere koppelingsconstanten, kan in een geschikt gekozen temperatuurgebied zeer goed worden benaderd door een uitdrukking die alleen de spin-spin correlatiefunctie voor naaste buren langs de eerstgenoemde roosteras bevat.

   Dit proefschrift, hoofdstuk IV.

3. Bij de gangbare uitspraak dat de Bethe-Peierls benadering voor een Ising-systeem op een gegeven rooster equivalent is met de exacte theorie voor een zgn. Bethe-rooster met hetzelfde coördinatiegetal als het oorspronkelijke rooster gaat men ten onrechte voorbij aan het feit dat het begrip thermodynamische limiet niet goed is gedefiniëerd voor een Bethe-rooster (behalve in het geval van de lineaire keten). Bedoelde uitspraak is slechts correct wanneer men uitgaat van het oneindige Bethe-rooster, en daarvoor thermodynamische grootheden per punt definiëert in termen van grafische reeksen.

   C. Domb, Advances in Physics 9 (1960) 283.

4. Bij de gebruikelijke afleiding van de relatie tussen de initiële susceptibiliteit enerzijds en alle spin-spin correlatiefuncties anderzijds voor een Ising-systeem boven de kritische temperatuur wordt impliciet verondersteld dat de oneindige sommatie over alle roosterpunten en de limiet $H \to 0$, waarbij H het uitwendige magneetveld is, verwisseld mogen worden.

   M.E. Fisher in: Lectures in Theoretical Physics, Vol. VII C, W.E. Brittin, ed., The University of Colorado Press (Boulder 1965), p. 72.

5. Voor een willekeurig Ising-systeem met slechts twee-deeltjes interacties geldt bij afwezigheid van een uitwendig magneetveld de volgende ongelijkheid voor spin-correlatiefuncties:

$$\langle\sigma_1\sigma_2\sigma_3\sigma_4\sigma_5\sigma_6\rangle - \langle\sigma_2\sigma_3\rangle\langle\sigma_1\sigma_4\sigma_5\sigma_6\rangle - \langle\sigma_2\sigma_4\rangle\langle\sigma_1\sigma_3\sigma_5\sigma_6\rangle -$$
$$- \langle\sigma_2\sigma_5\rangle\langle\sigma_1\sigma_3\sigma_4\sigma_6\rangle - \langle\sigma_3\sigma_4\rangle\langle\sigma_1\sigma_2\sigma_5\sigma_6\rangle - \langle\sigma_3\sigma_5\rangle\langle\sigma_1\sigma_2\sigma_4\sigma_6\rangle -$$
$$- \langle\sigma_4\sigma_5\rangle\langle\sigma_1\sigma_2\sigma_3\sigma_6\rangle + \langle\sigma_1\sigma_6\rangle\langle\sigma_2\sigma_3\sigma_4\sigma_5\rangle \leq 0.$$

6. Aan de hand van een eenvoudige topologische overweging kan men inzien dat de uitdrukking voor het kritische punt van het random-cluster model op het kwadratische rooster voor zeer grote waarden van de model-parameter κ van de vorm $1-p_c \sim \kappa^{-\frac{1}{2}}$ is, waarbij p de kans voor "realisatie" van een lijn is en p_c de kritische waarde van p.

7. Bij een stochastische wandeling op een lineaire keten waarin iedere lijn een kans p heeft om (permanent) defect te zijn, is het gemiddelde aantal stappen dat een wandelaar aflegt totdat hij voor het eerst een defecte lijn treft gelijk aan $p^{-2} - 1$; hij heeft dan gemiddeld van $p^{-1} - 1$ lijnen gebruik gemaakt.

8. Voor een systematische analytische bestudering van vibratiespectra van ongeordende ketens is de ontwikkeling van een theorie over stochastische produkten van niet-commuterende 2×2-matrices zeer gewenst, mede gezien het feit dat de (spaarzame) tot nu toe verkregen exacte resultaten alle zijn afgeleid via beschouwingen van zulke produkten.

9. De in de economie gebruikelijke methode van input-output tabellen en relaties is zeer geschikt voor een kwantitatieve studie van de fosfaatbalans in het oppervlaktewater.

Stellingen behorende bij het proefschrift van C.A.W. Citteur.

Leiden, 11 april 1973.

Aan mijn ouders

Aan Swinny

5. Voor een willekeurig Ising-systeem met slechts twee-deeltjes interacties geldt bij afwezigheid van een uitwendig magneetveld de volgende ongelijkheid voor spin-correlatiefuncties:

[illegible]

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7. Bij een stochastische wandeling op een lineaire keten waarin iedere lijn met kans p heeft [illegible] te zijn, is het gemiddelde aantal [illegible] dat een wandelaar [illegible] totdat hij voor het eerst een defecte lijn treft gelijk aan $p^{-2} - 1$. Hij heeft dan gemiddeld van $p^{-1} - 1$ lijnen gebruik gemaakt.

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9. De in de gangbare [illegible] methode [illegible] is zeer [illegible]

[illegible] C.A.W. Citteur.

Leiden, [illegible]

CONTENTS

INTRODUCTION AND SURVEY

Much attention has, especially during the last decade, been given to the way in which the critical behaviour of a physical system depends on the various properties of the Hamiltonian of the system. On the basis of such investigations one has arrived at the conclusion that critical behaviour depends on only a few "global" properties of a system, such as its dimensionality, the question as to whether the range of the interactions is finite or not (and if not, how the interactions decrease with the distance), and the symmetry properties of the Hamiltonian. In contrast to this, the details of the Hamiltonian, such as the values of the parameters indicating the strength of the interactions, seem to have no influence on the critical behaviour. One often refers to this feature of critical behaviour as universality.

In this thesis we are concerned with the status of this universality in the case of spin-$\frac{1}{2}$ Ising systems on extremely anisotropic Cartesian (= hypercubical) lattices, i.e. Cartesian lattices in which the coupling between nearest neighbours along one or more lattice axes is much weaker than that between nearest neighbours along the other lattice axes. In the case where the weak coupling is completely absent, the lattice in question, say L, consists in fact of uncoupled (identical) lattices of lower dimensionality, L'; the critical behaviour of the Ising system is then that which corresponds to these lattices of lower dimensionality L'. In the opposite case, however, no matter how small the weak coupling is, the system under consideration is of the original dimensionality. Universality would therefore imply that the critical behaviour is the same for all cases where the weak coupling is not completely absent, but changes discontinuously to that of L' at the moment at which the weak coupling has disappeared completely. In view of this fact the question as to how thermodynamic quantities go over into those for the lower-dimensional lattice in the limit of vanishing weak coupling, becomes an intriguing one.

From the investigations described in this thesis it appears that thermodynamic quantities of Ising systems on extremely anisotropic lattices display, depending on the conditions imposed, the critical behaviour characteristic for the original lattice L, or the critical behaviour characteristic for the lattice L' (or even an arbitrary intermediate behaviour). For any fixed choice of the values of the coupling constants J_i, characterizing the strength of the interactions between nearest neighbours along the respective lattice axes, only the first type of critical behaviour can occur. In speaking about universality one usually has this situation in mind; it follows that under this condition critical behaviour satisfies indeed the universality hypothesis. The other forms of critical behaviour (to which apparently universality does not apply) arise if we allow for variations not only in the temperature, but also, simultaneously, in the coupling constants. At first sight situations of the latter type may seem rather "unphysical". However, in cooling down a sample in a real experiment one might expect that the coupling constants change as a result of the variation in distance between the atoms of the sample accompanying the change in temperature. Furthermore, one might think of experiments in which simultaneously with a change in temperature the system is subject to a large compression or dilatation, resulting in a rather drastic change in the coupling constants. Variations in the coupling constants are therefore not as artificial as they might appear.

In addition to the considerations on universality given above there are other reasons why the investigation of Ising systems on extremely anisotropic lattices may be useful. As is well known, only a very limited number of closed analytic expressions for thermodynamic quantities of Ising systems is available, viz. only for systems on one- and two-dimensional lattices. For three-dimensional Ising systems no thermodynamic quantity is known in closed form; even an expression for the critical temperature in terms of the coupling constants is missing. One may hope, however, that in contrast with the case of general values of the coupling constants, it is possible to obtain analytic information for cases of extreme anisotropy, where the lattice L may be thought of as consisting of the lattices L', which are extremely weakly coupled to each other. If some thermodynamic quantities of L' are known in closed form, this could entail some analytic information for L in the corresponding case of extreme anisotropy. The results of our investigations contain indeed some progress in this direction, as will appear below.

In view of the smallness of one or more coupling constants for extremely anisotropic lattices, it is natural to investigate the thermodynamic quantities of such lattices by means of power series in these constants or, more generally, in variables vanishing with them. The coefficients in such series will have the general property that they can be expressed completely in terms of multiple-spin correlation functions on the set of uncoupled lattices L'. We therefore expect that the coefficients display a singular behaviour at the critical temperature of L'; however, this temperature is not the critical temperature of the lattice L. It follows that if the series is truncated, the resulting function has not the critical behaviour characteristic for L; this is displayed only by the full series - whose calculation is beyong our power in general. In spite of this, it is to be expected that the critical properties of the coefficients will become more and more important as the critical temperature of L approaches that of L', which takes place in the limit of extreme anisotropy. On account of this it is desirable to obtain information at least on the dominant singular part of each coefficient.

In this thesis we have investigated series of the type mentioned above for the zero-field susceptibility; as is well known, this quantity can be studied very accurately by means of series expansions. In view of the foregoing we have restricted our attention mainly to the dominant singular part of the coefficients; the quantity which results if we replace each coefficient by its most singular part has been called the <u>leading-order term</u> of the susceptibility. In addition to the properties of this leading-order term we have studied, by means of this quantity, the asymptotic form of the equation for the critical temperature, especially of the simple cubic lattice, in cases of extreme anisotropy.

We start in chapter I by considering the ferromagnetic quadratic lattice. We cannot calculate the leading-order term in closed form, but only its first few terms. From the general structure we conclude that the singular behaviour has various aspects: depending on the conditions imposed the leading-order term has a critical behaviour of the same type as that of the susceptibility of the linear chain (which in this case is the lattice L'), or a critical behaviour which, within the limits of accuracy, is of the same type as that of the susceptibility of the quadratic lattice, or any intermediate behaviour. Furthermore, it turns out that we can obtain from the first few terms, again within the limits of accuracy, the asymptotic form of the well-known expression for the critical temperature of the quadratic lattice. We usually write the dependence of the critical temperature on the coupling constants in the form of an equation in the variables $t_i = \tanh J_i/kT$, and call this equation the <u>critical equation</u> of the lattice under consideration. The critical equation for the quadratic lattice is then a simple bilinear equation in t_1 and t_2.

In chapter II we consider the ferromagnetic simple cubic lattice in which the coupling constants along two lattice axes are much smaller than the coupling constant along the remaining lattice axis; the investigation is based on a straightforward extension to the simple cubic lattice of the method developed in chapter I. Similar conclusions as in chapter I can be drawn on the critical behaviour of the leading-order term of the susceptibility. On the analogy of the fact that for the quadratic lattice the asymptotic critical equation can be obtained from the leading-order term of the susceptibility, we assume that this is also true for the present case, where the critical equation is not known explicitly. From the (numerical) results it then appears that the critical equation for the simple cubic lattice is certainly not a trilinear equation in the variables t_1, t_2 and t_3, but most probably a much more complicated equation. Expressing our results in a geometrical language we can say that the critical surface in the t_1, t_2, t_3-space corresponding to this equation behaves as a cone rather than as a plane in the immediate neighbourhood of the points (1,0,0), (0,1,0) and (0,0,1).

In chapter III the method followed in chapters I and II for developing the leading-order term of the susceptibility is made much more efficient and less cumbersome by means of the extension to anisotropic lattices of a formula, originally proposed by Sykes and derived later on by Nagle and Temperley, for the susceptibility of isotropic lattices.

Chapter IV is devoted to antiferromagnetic Ising systems on Cartesian lattices in which the coupling constant along one lattice axis is (in absolute value) much larger than those along the other lattice axes. The generalized formula of Sykes just mentioned turns out to form an ideal starting point here, in that it not only enables us to obtain numerical information more easily than before, but also leads us in a straightforward way to some analytic properties of the leading-order term; it is not clear how these properties could be established by means of other methods. More specifically, we can derive from the generalized formula a simple relation between the leading-order term of the susceptibility and the (similarly defined) leading-order term of the quantity $1 + \langle\sigma_1\sigma_2\rangle$ - where $\langle\sigma_1\sigma_2\rangle$ is the two-spin correlation function for nearest neighbours along the lattice axis with the large coupling constant - valid in the case where this constant is negative. Using this relation we can derive a closed form for the leading-order term of the antiferromagnetic susceptibility of the quadratic lattice in this case. Similar conclusions as in chapters I and II can be drawn on the critical behaviour of the leading-order term of the susceptibility for all antiferromagnetic cases.

In chapter V we study the Curie temperature of the simple cubic lattice in which the coupling constants along one or two lattice axes are very small in comparison with the remaining coupling constant(s). The results obtained and conjectured, which are analytic in nature, are in agreement with predictions from scaling theory. They imply that the above-mentioned critical surface touches the coordinate planes; this feature is consistent with the conical behaviour of the surface near the points (1,0,0), (0,1,0) and (0,0,1). On the basis of these results an explicit form for the asymptotic critical equation near (1,0,0) is proposed, which is in the best possible agreement with the numerical results found in chapter II.

equation for the simple cubic lattice is [illegible] the variables t_1, t_2 and t_3, but most probably a much more complicated equation. Expressing our results in a geometrical language we can say that the critical surface in the t_1,t_2,t_3-space corresponding to this equation behaves as a cone rather than as a plane in the immediate neighbourhood of the points (1,0,0), (0,1,0) and (0,0,1).

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CRITICAL BEHAVIOUR OF ISING SYSTEMS ON EXTREMELY ANISOTROPIC LATTICES

I. The quadratic lattice (ferromagnetic case)

Synopsis

The critical behaviour of the magnetic susceptibility χ of a spin-$\frac{1}{2}$ Ising system on a quadratic lattice in which the coupling constant along one of the lattice axes, J_2, is very small in comparison with the coupling constant along the other axis, J_1, is investigated on the basis of the series expansion of χ in the variables $t_1 = \tanh(J_1/kT)$ and $t_2 = \tanh(J_2/kT)$. It is shown that for $t_2 \ll 1$, $1-t_1 \ll 1$ the reduced susceptibility $\chi' \equiv (kT/\mu^2)\chi$ behaves as

$$\chi'(t_1,t_2) \approx \chi_0'(t_1,t_2) \equiv \frac{1}{1-t_1} \sum_{n=0}^{\infty} b_{n0} \left(\frac{t_2}{1-t_1}\right)^n ,$$

where the b_{n0} are constants. In the limit $(t_1,t_2) \to (1,0)$, taken in such a way that the series $\Sigma_n b_{n0} [t_2/(1-t_1)]^n$ converges, χ_0' diverges through its first factor as the susceptibility of a linear Ising chain. On the other hand, the two-dimensional nature of the system is displayed by the behaviour of χ_0' in the limit $(t_1,t_2) \to (\tau,b^{-1}(1-\tau)) \neq (1,0)$, where b^{-1} is the radius of convergence of the power series $\Sigma_n b_{n0} x^n$. Numerical evidence suggests that in this limit χ_0' diverges as $[1-bt_2/(1-t_1)]^{-p}$ with $b = 2.02 \pm 0.03$ and $p = 1.77 \pm 0.03$. The exact value of b should be 2, as can be seen from the equation for the critical temperature in terms of the variables t_1 and t_2; that of p is most probably equal to the critical exponent of χ for the isotropic lattice, $7/4$.

§ 1. Introduction

In the literature on the Ising model most attention has hitherto been given to the properties of isotropic Ising systems. To be sure, closed expressions have been derived for a number of thermodynamic quantities of anisotropic Ising systems, but their derivation is a trivial generalization of that valid for isotropic systems. For approximative theories, however, such as those of Bethe and Peierls, and of Kikuchi, such a generalization, although equally trivial, seems not to have been carried out. Series expansions for thermodynamic quantities have, with a few exceptions [1) 2)], also been derived for isotropic lattices only.

It is the aim of this and subsequent papers to study some properties of anisotropic Ising systems more in detail. We shall restrict ourselves to spin-$\frac{1}{2}$ Ising systems with ferromagnetic nearest-neighbour interactions on d-dimensional Cartesian lattices. By a d-dimensional Cartesian lattice we understand a lattice consisting of all points in d-dimensional Euclidean space with integral Cartesian coordinates, and by nearest neighbours two lattice points which are a unit distance apart. If we connect each pair of nearest neighbours by a line, the Hamiltonian of a finite Ising system as referred to above can be written as

$$\mathcal{H} = -\tfrac{1}{2}\sum_{i,j} J_{ij}\,\sigma_i\,\sigma_j, \tag{1}$$

where $i,j = 1,\ldots, N$ label the points of the system, $\sigma_i (= \pm 1)$ is the spin variable associated with the point i, and

$$J_{ij} = \begin{cases} J_r & \text{if the points } i \text{ and } j \text{ are connected by a line parallel to} \\ & \text{the } r^{\text{th}} \text{ lattice axis } (r = 1,\ldots, d), \\ 0 & \text{otherwise;} \end{cases} \tag{2}$$

the coupling constants J_r are supposed to be non-negative throughout this paper.*)

We shall pay special attention to extremely anisotropic systems, i.e. to the limiting cases where the values of one or more parameters J_r are extremely small in comparison with the other ones, and thus study the way in which the properties of an Ising system on a d-dimensional Cartesian lattice reduce to those of a system on a lattice of lower dimensionality as some of the J_r vanish. Earlier, Weng, Griffiths and Fisher [2)] discussed this limit of extreme anisotropy.

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*) Cases where one or more of the J_r are negative will be discussed in a subsequent paper.

They analysed the way in which the Curie temperature T_c of an Ising system on a d-dimensional Cartesian lattice vanishes as all coupling constants except one, say J_1, go to zero simultaneously. Their results are given in terms of the ratio $\eta = (J_2+\ldots+J_d)/J_1$. It turns out that in this respect the asymptotic behaviour of T_c is essentially the same for all d, viz.:

$$\frac{kT_c}{2J_1} = \left[\log\eta^{-1} - \log\log\eta^{-1} + \mathcal{O}(1)\right]^{-1} \qquad (\eta\to 0); \tag{3}$$

moreover, the same asymptotic form was found in mean-field theories such as the Bethe-Peierls approximation. As we shall see, differences come in if we describe the properties of the system in terms of the variables $t_r = \tanh\beta J_r$ ($\beta = 1/kT$), which enter in a natural way if the Ising problem is considered as a combinatorial problem with respect to the lattice under consideration; obviously, $0 \leq t_r \leq 1$ for all r. Thermodynamic functions can then be written as multiple power series in all variables t_r. If, for given values of the coupling constants J_r, there is one, and only one, temperature $T = T_c(J_1,\ldots,J_d)$ at which some (or all) of the thermodynamic quantities are singular, the equation defining this <u>critical temperature</u>, to be called the <u>critical equation</u> henceforth, can be written in the form

$$\Delta_d(t_1,\ldots,t_d) = 0. \tag{4}$$

Obviously, the function Δ_d is not uniquely determined: if the function Δ_d has the required property, so has the function $f(\Delta_d)$, where f is an arbitrary function with $f(0) = 0$ (e.g. $f(\Delta_d) = \Delta_d^2$). We suppose that a suitable choice for Δ_d has been made; for the quadratic lattice [3], e.g., a good choice is

$$\Delta_2(t_1,t_2) = 1 - t_1 - t_2 - t_1t_2\ . \tag{5}$$

It is a striking property of those thermodynamic quantities of the Ising system on a quadratic lattice which are known in closed form (e.g. the spontaneous magnetization M, the specific heat in zero magnetic field $C_{H=0}$, and the spin-spin correlation functions $\langle\sigma_i\sigma_j\rangle$ for two spins i and j whose positions differ by a vector parallel either to one of the lattices axes or to one of the main diagonals of the lattice) that their critical behaviour can be described in terms of the single function $\Delta_2(t_1, t_2)$ in the following sense:

$$\left.\begin{aligned} M(t_1,t_2) &\simeq a(t_{1c}, t_{2c})|\Delta_2|^{1/8}, \\ C_{H=0}(t_1,t_2) &\simeq b(t_{1c}, t_{2c}) \log|\Delta_2|, \\ \langle\sigma_i\sigma_j\rangle - \langle\sigma_i\sigma_j\rangle_c &\simeq c_{ij}(t_{1c}, t_{2c})\, \Delta_2 \log|\Delta_2|, \end{aligned}\right\} \tag{6}$$

$$\text{for } (t_1,t_2) \to (t_{1c}, t_{2c}), \text{ where } t_{1c}, t_{2c} \text{ satisfy: } \begin{cases} \Delta_2(t_{1c}, t_{2c}) = 0, \\ t_{1c} \neq 0, \\ t_{2c} \neq 0 . \end{cases}$$

For M, the limit has to be taken from the low-temperature side, i.e. from the side where Δ_2, as given by eq. (5), is negative; for $C_{H=0}$ and the correlation functions it may be taken from both sides.

One might conjecture that this is a more general phenomenon, viz. that for any Ising system the critical behaviour of each thermodynamic quantity can be described in terms of the function $\Delta_d(t_1, \ldots, t_d)$ alone, at least for sufficiently general values of $t_1, \ldots, t_d$ (we know that the critical behaviour may change as soon as one or more of the t_r vanish). From this point of view, Δ_d seems to be the most adequate variable in terms of which to express critical behaviour; the fact that Δ_d is not uniquely defined need not present a serious difficulty. Determining Δ_d as a function of $t_1, \ldots, t_d$ is, therefore, a natural first step in a theoretical analysis of the properties of the Ising model in d dimensions.

Since Δ_d is known for d=2 (cf. eq. (5)), the first case to be investigated is d=3. Since the problem of completely determining $\Delta_3(t_1,t_2,t_3)$ looks prohibitive at present, we shall satisfy ourselves with a study of the asymptotic behaviour of this function for extremely small values of one or two parameters t_r. We shall try to determine this asymptotic behaviour by means of a study of series expansions, which seem to form the only tool available at present. Experience has shown that the most reliable results on critical points are found from a high-temperature expansion of the initial magnetic susceptibility χ; therefore, we shall concentrate on this quantity. In order to get acquainted with series expansions in more than one variable, and to get an idea about the reliability of the method we shall in this paper investigate the series expansion of the susceptibility of the anisotropic quadratic lattice. The critical equation for this lattice is, according to (4) and (5):

$$1 - t_1 - t_2 - t_1t_2 = 0. \tag{7}$$

Substituting $t_2 = 0$ we get the critical equation for the linear chain:

$$1 - t_1 = 0, \tag{8}$$

which expresses the well-known fact that for this lattice T=0 may be considered as the critical temperature. Obviously, we may put

$$\Delta_1(t_1) = 1 - t_1 = \Delta_2(t_1, 0).$$

For $0 < t_2 \ll 1$, eq. (7) reduces to

$$1 - t_1 \approx 2t_2 \qquad (t_2 \to 0). \tag{9}$$

It is this asymptotic result which we shall have to find from the analysis of the series expansion of χ. Apart from this result, the series expansions are interesting in themselves as they provide new information on the susceptibility χ, for which no expression in closed form is known except for the trivial case d=1.

To show the difference between exact and approximate results referred to above, we begin by briefly discussing, in § 2, the extension of the Bethe-Peierls and the Kikuchi approximation to anisotropic systems. They will be shown to lead to an equation of the same form as eq. (9), the only difference lying in the numerical factor in the right-hand side.

In § 3 we present the general formalism for finding the series expansion of the susceptibility in terms of graphs on the lattice. Although this procedure is more or less standard (see, e.g., ref. 4), the treatment of extremely anisotropic lattices makes it desirable to discuss its various steps in detail.

In § 4 the terms of the series are classified in a way appropriate to the special case of extreme anisotropy. To this end, the series is rewritten as a power series in t_2 with coefficients depending on t_1, in view of the fact that t_2 is very small. Each of these coefficients turns out to be the sum of contributions which diverge for $t_1 \to 1$. We assume that for determining the asymptotic behaviour of Δ_2 it is sufficient to take into account only the most divergent contributions. This reduces the actual calculation, which is given in § 5, very much, although the calculation of as few as five terms is still laborious.

In the last section the coefficients are analysed by means of the ratio method and its extensions. The assumption mentioned before is hereby confirmed, which gives a certain confidence in its applicability in other cases, where Δ_d

is not known in advance.

For brevity we shall use the following nomenclature. The axes of the quadratic lattice are called the x- and y-axis. Any concept which is related to a particular axis is named after that axis, e.g. a pair of nearest neighbours which are connected by a line parallel to the x-axis is called a pair of x-neighbours, the line itself an x-line, a chain of subsequent x-lines is called an x-chain etc. In addition, we shall use the terms horizontal (vertical) for: parallel to the x(y)- axis; right (left), referring to the direction of the positive (negative) x-axis; upper, upward (lower, downward), referring to the direction of the positive (negative) y-axis.

§ 2. Approximative methods for the anisotropic quadratic lattice

a) The Bethe-Peierls approximation

The method used to derive the Bethe-Peierls approximation for an anisotropic quadratic lattice is a direct generalization of the original method developed for isotropic lattices by Bethe [5]. Accordingly, only the main points will be presented.

Consider an arbitrary lattice point together with its four neighbours. Let $P(\sigma; n_1, n_2)$ be the probability that the spin variable of the central point has the value σ ($\sigma = \pm 1$) and that the spin variables of n_1 of its two x-neighbours and n_2 of its two y-neighbours have the value +1. In evaluating this probability, the interaction between the central spin and its neighbours is exactly taken into account, whereas the influence of the rest of the lattice on the neighbours is approximately expressed with the aid of two positive parameters z_1 and z_2 as follows:

$$P(+1; n_1, n_2) = C\binom{2}{n_1}\binom{2}{n_2}\exp\left[\beta J_1(2n_1-2)\right]\exp\left[\beta J_2(2n_2-2)\right] z_1^{n_1} z_2^{n_2}, \tag{10a}$$

$$P(-1; n_1, n_2) = C\binom{2}{n_1}\binom{2}{n_2}\exp\left[\beta J_1(2-2n_1)\right]\exp\left[\beta J_2(2-2n_2)\right] z_1^{n_1} z_2^{n_2}, \tag{10b}$$

where C is a normalizing factor.

The values of the parameters z_1 and z_2 are determined by the requirement that the total probability for the spin variable to have the value +1 be the same for the central point and for its x- and y-neighbours. This requirement leads to the equation

$$z_1 = \frac{z_1 \exp(2\beta J_1) + 1}{z_1 + \exp(2\beta J_1)} \left(\frac{z_2 \exp(2\beta J_2) + 1}{z_2 + \exp(2\beta J_2)}\right)^2 , \tag{11}$$

and to the equation obtained from (11) by interchanging the indices 1 and 2.

By eliminating z_2 one obtains an equation for z_1. For $\beta < \beta_c$, with β_c determined by

$$1 - \exp(-2\beta_c J_1) - \exp(-2\beta_c J_2) = 0, \tag{12}$$

this equation has only the trivial solution $z_1 = 1$. For $\beta > \beta_c$ it has three roots: $z_1 = 1$, $z_1 = z$, say, $z_1 = z^{-1}$ $(z > 1)$. The last two roots turn out to correspond to stable solutions with opposite values of the magnetization, the first root to an unstable solution. Evidently, eq. (12) is the equation for the Curie temperature in the Bethe - Peierls approximation. Written in terms of t_1, t_2 it reads

$$1 - t_1 - t_2 - 3t_1t_2 = 0 , \tag{13}$$

which for $J_2 \ll J_1$ reduces to

$$1 - t_1 \simeq 4t_2 \qquad (t_2 \to 0). \tag{14}$$

b) The Kikuchi approximation

An essential feature of the Bethe-Peierls approximation is the fact that the only correlations between spins which are exactly taken into account are those between two neighbouring spins. Kikuchi [6] has shown how this approximation can be improved by taking also into account the correlations between larger groups of spins, in particular those between all spins of a set of atoms forming the smallest possible polygon on the lattice. For the quadratic lattice, where this polygon is a square, a straightforward generalization of Kikuchi's calculation to the anisotropic case can be shown to yield the following critical equation:

$$1 - t_1 - t_2 - t_1^2 - t_1t_2 - t_2^2 + t_1^3 + t_1^2t_2 + t_1t_2^2 + t_2^3 + t_1^3t_2 - 2t_1^2t_2^2 + t_1t_2^3 = 0. \tag{15}$$

For the isotropic case it gives $t_c = \frac{-1 + \sqrt{17}}{8}$ [6]; for $J_2 \ll J_1$ the asymptotic form of the equation is

$$1 - t_1 \simeq (1 + \sqrt{2})t_2 \qquad (t_2 \to 0). \tag{16}$$

Now we can easily see why, as was stated in the introduction, the approximations discussed here and the exact equation (7) all yield the same asymptotic behaviour of T_c in terms of $\eta \equiv J_2/J_1$. Both the exact equation (9) and the approximative equations (14) and (16) are of the form

$$1 - t_1 \simeq bt_2 \qquad (t_2 \to 0), \tag{17}$$

in which b is a non-vanishing constant. Expressing (17) in terms of J_1, η and kT_c, letting η, and hence T_c, go to zero while J_1 is kept constant, and retaining the leading terms only, one finds indeed eq. (3). Since in this equation the constant b does not appear in leading orders, the asymptotic behaviour of T_c, thus described, is independent of the approximation made.

§ 3. High-temperature series expansion for the susceptibility (general method).

As is well known [7], the initial magnetic susceptibility per spin of the Ising model in the thermodynamic limit can, for $T > T_c$, be written as

$$\chi = \frac{\mu^2}{kT}\left(1 + \sum_{j\neq 1} \langle\sigma_1\sigma_j\rangle\right), \tag{18}$$

in which 1 is an arbitrarily chosen point of the infinite lattice L and j runs over the remaining points; μ is the magnetic moment of each spin and $\langle\ \rangle$ denotes the thermodynamic limit of a thermal average in the absence of an external magnetic field.

Obviously, the critical behaviour of χ is contained in the second term of the right-hand side of equation (18), to which we shall restrict our attention. The evaluation of the sum $\Sigma_{j\neq 1}\langle\sigma_1\sigma_j\rangle$ goes as follows. First, we consider a finite lattice of N points with periodic boundary conditions, L_N, of which, for simplicity, all coupling constants are supposed to have the same magnitude J. The lattice, together with the lines connecting nearest neighbours, forms a graph, also to be denoted by L_N. By a partial graph of L_N one understands a subgraph of L_N containing all points of L_N, by a line component of L_N a maximal connected subgraph containing at least one line. The number of lines incident with a point is called the valence of the point.

In L_N the thermal average $\langle\sigma_1\sigma_j\rangle_N$ is by definition:

$$\langle\sigma_1\sigma_j\rangle_N = \frac{\sum_{\{\sigma\}} \sigma_1\sigma_j \exp[-\beta E(\{\sigma\})]}{\sum_{\{\sigma\}} \exp[-\beta E(\{\sigma\})]}, \tag{19}$$

where the sums run over the 2^N spin configurations $\{\sigma\}$ of the system and $E(\{\sigma\})$ is the energy of the configuration $\{\sigma\}$. Using standard combinatorial methods [8] one can rewrite $\Sigma_{j\neq 1}\langle\sigma_1\sigma_j\rangle_N$ as follows:

$$\sum_{j\neq 1} \langle\sigma_1\sigma_j\rangle_N = \frac{\sum_{G_2} t^{|G_2|}}{\sum_{G} t^{|G|}}, \quad t = \tanh\beta J, \tag{20}$$

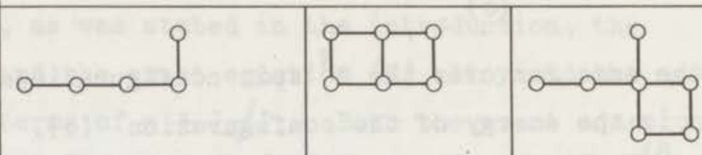
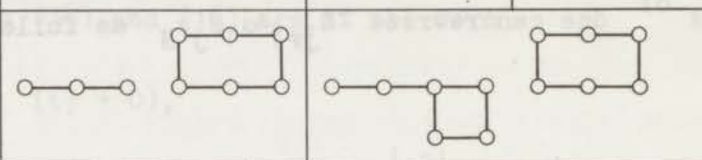
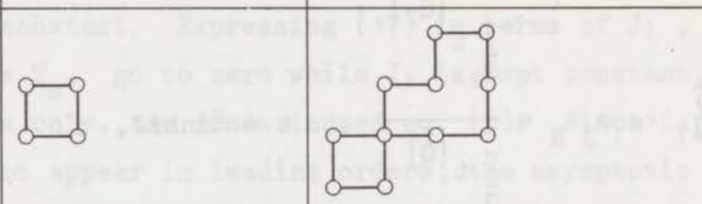
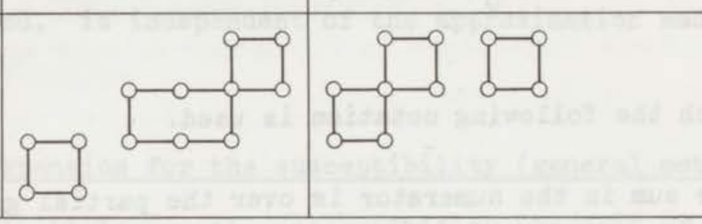
in which the following notation is used.

i) The sum in the numerator is over the partial graphs G_2 of L_N in which each point except the fixed point 1 and the variable point j has even valence (possibly zero); $|G_2|$ is the number of lines in G_2. Since 1 and j are the only points of odd valence, they belong to the same line component of G_2, according to a well-known theorem by Euler [9].

ii) The sum in the denominator is over the partial graphs G of L_N in which each point has even valence.

Let us denote by [2], [2 0], [2 0 0], ... the contributions to $\Sigma_{G_2} t^{|G_2|}$ of those partial graphs G_2 which contain one, two, three, ... line components; the figure 2 represents the component containing the two points of odd valence, the figures 0 the other components. Similarly, we split the sum $\Sigma_G t^{|G|}$ into terms [ ], [0], [0 0], ... , where the first term represents the contribution from the subgraph containing no lines, and hence no line components. The types of partial graphs yielding the contributions [2], [2 0], ... ; [0], [0 0], ... will be denoted by 2, 2 0, ... ; 0, 0 0, ... Some examples for the quadratic lattice are listed in table I; here, as in the following, the isolated vertices of G_2 and G have been omitted.

TABLE I
Some partial graphs on the quadratic lattice of the types 2, 20, 0 and 00.

types of partial graphs	examples
2	
2 O	
O	
O O	

With this notation we can write equation (20) as follows:

$$\sum_{j\neq 1} \langle\sigma_1\sigma_j\rangle_N = \frac{[\,2\,]+[\,2\;0\,]+[\,2\;0\;0\,]+\dots}{[\;\;]+[\,0\,]+[\,0\;0\,]+\dots} . \tag{21}$$

The division can be carried out formally; it is convenient to do this step by step, increasing at each step the number of line components involved by one. The successive terms of $\Sigma_{j\neq 1}\langle\sigma_1\sigma_j\rangle_N$ obtained in this way, and described more fully below, will be denoted by A_1, A_2, ... :

$$\sum_{j\neq 1} \langle\sigma_1\sigma_j\rangle_N = \sum_{k=1}^{\infty} A_k . \tag{22}$$

1) The first step consists in dividing [2] by [ ]; since [ ] equals 1, this yields:

$$A_1 = [\,2\,] . \tag{23}$$

So A_1 contains the contributions to $\Sigma_{j\neq 1}\langle\sigma_1\sigma_j\rangle_N$ from partial graphs containing one single line component with two points of odd valence.

2) The next step of the division yields

$$A_2 = [2\ 0] - [2]\cdot[0]. \tag{24}$$

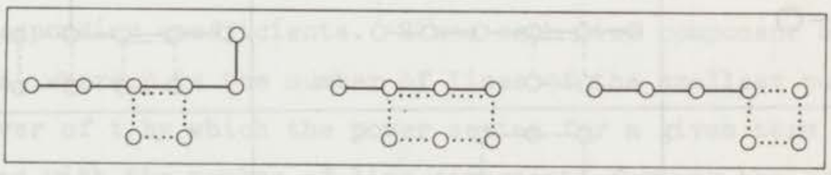
Now $[2]\cdot[0]$ is the sum of $t^{|G_2|}t^{|G|}$ over all combinations of a graph G_2 and a graph G both containing exactly one line component. Those combinations where G_2 and G have no points (and, hence, no lines) in common sum up to $[2\ 0]$. In addition, there are combinations where G_2 and G have at least one point in common. It follows from (24) that only these are not cancelled in A_2; denoting the sum of $t^{|G_2|}t^{|G|}$ over these combinations of G_2 and G by $[2\text{-}0]$ we have:

$$A_2 = -[2\text{-}0]. \tag{25}$$

Some examples are given in table II.

TABLE II

Some combinations of one line component of type 2 (indicated by drawn lines) and one of type 0 (indicated by dotted lines), contributing to [2-0].



3) Using the relation $[2]\cdot[0] - [2\ 0] = [2\text{-}0]$, the contribution to $\Sigma_{j\neq 1}\langle\sigma_1\sigma_j\rangle_N$ arising from combinations of three line components can be written as

$$A_3 = [2\ 0\ 0] - [2]\cdot[0\ 0] + [2\text{-}0]\cdot[0]. \tag{26}$$

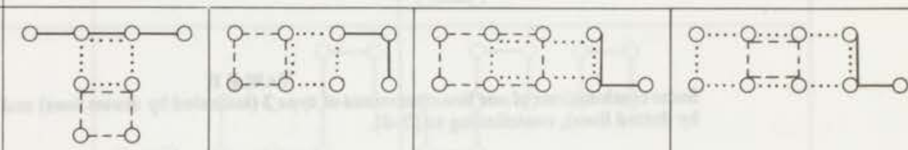
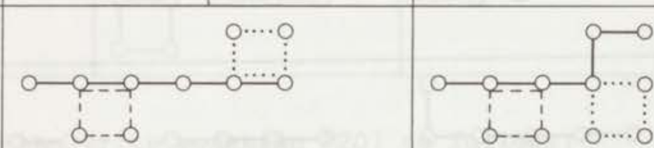
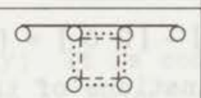
The last two terms in (26) can be analysed in the same way as the term $[2]\cdot[0]$ in (24):

$$[2]\cdot[0\ 0] = [2\ 0\ 0] + [2\text{-}0\ 0] + [0\text{-}2\text{-}0], \tag{27}$$

$$[2\text{-}0]\cdot[0] = [2\text{-}0\ 0] + [2\text{-}0\text{-}0] + 2[0\text{-}2\text{-}0] + 2[\overset{2}{0\text{-}0}] + [2\text{-}0^2]. \tag{28}$$

Again, linked figures represent line components having points in common. Combinations (i.e. sets) of several linked line components we shall call <u>overlappings</u>. The types of overlappings will be denoted by the same symbols as their contributions, with the omission of the brackets, e.g. 2-0-0, $2\text{-}0^2$. The last term in (28) represents those combinations of 2-0 and 0 in which the two partial graphs of type 0 are identical. The factor 2 in front of [0-2-0] arises from the fact that an overlapping of the type 0-2-0 can be built up in two ways from an overlapping of type 2-0 and a line component of type 0; similarly for $\begin{smallmatrix}2\\ /\,\backslash\\ 0-0\end{smallmatrix}$. Some examples of overlappings contributing to (27) and (28) are given in table III.

TABLE III
Some overlappings of one line component of type 2 (indicated by drawn lines) and two of type 0 (indicated by dashed and dotted lines). The last two examples for 2-0-0 illustrate the fact that an overlapping cannot be uniquely characterized by the (multi)graph consisting of its points and lines.

type of overlapping	examples
2-0-0	
0-2-0	
$\begin{smallmatrix}2\\ /\,\backslash\\ 0-0\end{smallmatrix}$	
$2\text{-}0^2$	

Combining (26), (27) and (28) we obtain

$$A_3 = [2\text{-}0\text{-}0] + [0\text{-}2\text{-}0] + 2\left[\begin{smallmatrix}2\\ /\,\backslash\\ 0-0\end{smallmatrix}\right] + [2\text{-}0^2]. \tag{29}$$

Similar expressions can be derived for A_4, A_5, etc. Since line components of type 2 and overlappings of one line component of type 2 and one or more line components of type 0 play a central role in this method of calculating χ, we shall introduce a common name for them, viz. <u>2-graphs</u>. Observe that in general a 2-graph is not a graph but a combination (i.e. a set) of graphs. Obviously, A_k stands for the contribution to the quantity $(kT/\mu^2)\chi$ from all 2-graphs consisting of k line components.

Each term in the right-hand side of the equations (23), (25), (29), etc. stands for the total contribution from all 2-graphs of a given type on L_N, the contribution from an individual 2-graph being equal to t^n, where n is the sum of the numbers of lines of the constituting (overlapping) line components. Because of the finiteness of L_N the terms [2], [2-0], etc. are polynomials in t; the coefficient of each power t^n in these polynomials becomes independent of N for sufficiently large values of N. This enables us to proceed to the thermodynamic limit by simply taking into account all possible 2-graphs of a given type on the <u>infinite</u> lattice L, irrespective of their size. The terms [2], [2-0], etc. then become infinite power series in t. Combining (18), (22), (23), (25) and (29) we find the following expression for the thermodynamic limit of the reduced susceptibility $\chi' \equiv (kT/\mu^2)\chi$:

$$\chi' - 1 = \sum_{j\neq 1} \langle\sigma_1\sigma_j\rangle = [2] - [2\text{-}0] + [2\text{-}0\text{-}0] + [0\text{-}2\text{-}0] + 2[\substack{2\\ \diagup\;\diagdown\\ 0-0}] + [2\text{-}0^2] + \ldots, \tag{30}$$

in which each term stands for the contribution from all 2-graphs of a given type on L. The coefficient of t^n in the series expansion for χ' is found by selecting from each of the separate power series for [2], [2-0], etc. the term with t^n and adding the corresponding coefficients. Since each line component of type 0 has at least p lines, where p is the number of lines of the smallest polygon on the lattice, the power of t by which the power series for a given term in (30) starts, increases with the number of line components forming the corresponding 2-graph. Therefore, in calculating the coefficient of t^n in the series expansion of χ' only a finite number of terms in (30) has to be taken into account.

The generalization of the procedure sketched above to anisotropic lattices is straightforward: quantities like $t^{|G|}$ are replaced by $\Pi_{r=1}^{d} t^{|G|_r}$ where the product runs over the d lattice axes, $t_r = \tanh\beta J_r$, and $|G|_r$ is the number of lines parallel to the r^{th} lattice axis in the graph G.

§ 4. Susceptibility of the extremely anisotropic quadratic lattice

We shall now apply the method introduced in the previous section to the anisotropic quadratic lattice. There, the reduced susceptibility χ' can be written as a double power series in t_1 and t_2 :

$$\chi' = 1 + \sum_{\substack{n_1,\, n_2=0\\ (n_1+n_2\geq 1)}}^{\infty} a_{n_1 n_2} t_1^{n_1} t_2^{n_2} \; . \tag{31}$$

Performing the summation over n_1 for each value of n_2 we obtain a single power series in t_2 with coefficients depending on t_1 :

$$\chi' = 1 + \sum_{n=0}^{\infty} a_n(t_1)t_2^n \ , \tag{32}$$

$$a_n(t_1) = \sum_{\substack{n_1=0 \\ (n_1+n \geq 1)}}^{\infty} a_{n_1 n} t_1^{n_1} \ . \tag{33}$$

In equation (32) $a_n(t_1)t_2^n$ is the contribution to χ' from 2-graphs containing an arbitrary number of x-lines and a fixed number n of y-lines (with the restriction that the total number of lines is at least 1). So $a_0(t_1)$ consists of the contributions from all 2-graphs containing no y-line and at least one x-line. These 2-graphs are line components consisting of a chain of subsequent x-lines (x-chain), running from the point 1 to the right or to the left, together with the points incident with these lines; their contributions are readily evaluated:

$$a_0(t_1) = 2 \sum_{n_1=1}^{\infty} t_1^{n_1} = \frac{2t_1}{1-t_1} \ . \tag{34}$$

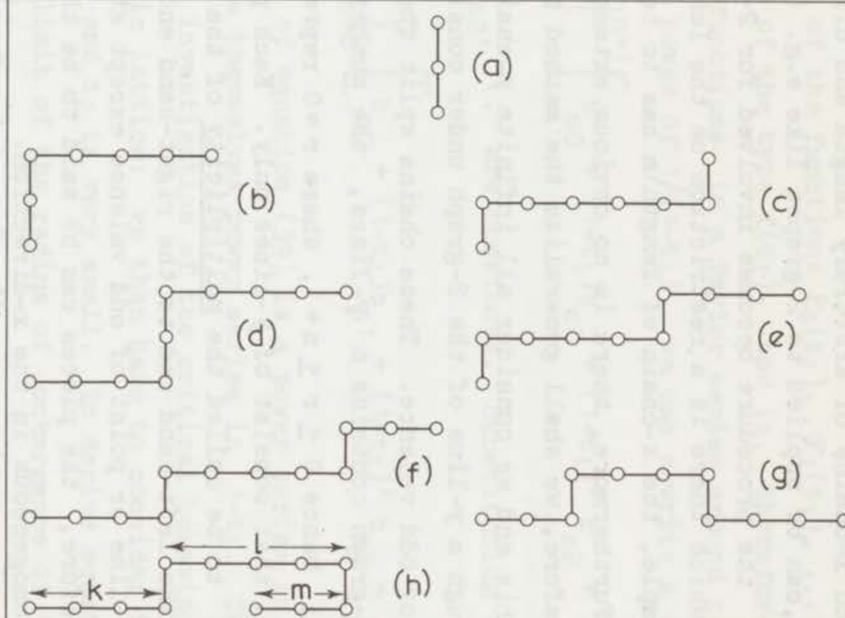
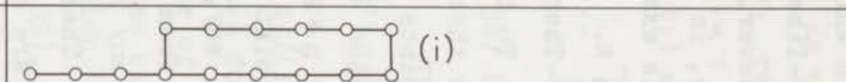
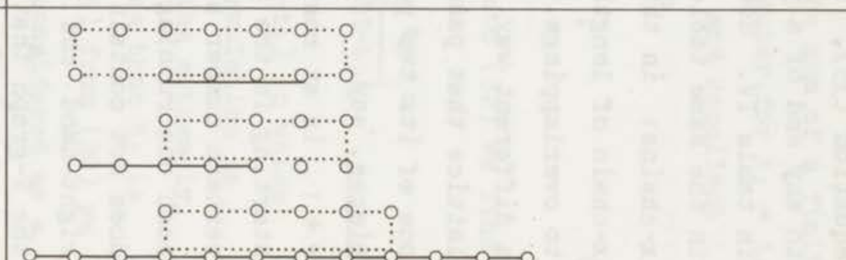
The 2-graphs contributing to $a_1(t_1)$ are line components containing one y-line. To each of the two ends of this y-line we may, independently, attach or not attach an x-chain, running to the right or to the left. Consequently,

$$a_1(t_1) = 2\left(1 + \frac{2t_1}{1-t_1}\right)^2 \ , \tag{35}$$

the factor 2 in front arising from the possibility that, viewed from the point 1, the y-line goes into the +y or into the -y direction. Since overlappings contain at least two y-lines they do not contribute to $a_0(t_1)$ and $a_1(t_1)$.

The 2-graphs contributing to $a_2(t_1)$ fall into three categories: single line components without points of valence ≥ 3 (self-avoiding walks), line components with one point of valence 3, and overlappings of a rectangle and an x-chain (see table IV).

TABLE IV

some 2-graphs contributing to $a_2(t_1)$		
type of 2-graphs	category of 2-graphs	examples
2	self-avoiding walks containing: no x-chain; one x-chain; two x-chains; three x-chains	
2	line component with one point of valence 3	
2-O	overlappings of a rectangle and an x-chain	

The straightforward generalization of the method followed to derive equation (35), in which x-chains of arbitrary length and direction are added to any end of a y-line,can be applied to 2-graphs like e.g. (d), (e) and (f) in table IV. However, the procedure becomes involved for 2-graphs like (h) in the same table, in which there is a restriction on the length of one or more x-chains: in this example, the x-chain of length m has to be shorter than the x-chain of length ℓ. Furthermore, there is no obvious extension of this method to overlappings. Therefore, we shall generalize the method of calculation in a different way. To this end we consider all infinite y-chains of the quadratic lattice that pass through a y-line of the 2-graph under consideration or through one of its two points of odd valence. These chains split the 2-graph into r <u>pieces</u>, say. If the 2-graph contains n y-lines, the number of these chains, $r+1$, is at most $n+2$; hence $0 \leq r \leq n+1$, where $r=0$ represents the two walks starting in the point 1 that consist of y-lines only. Each piece consists of a certain number μ, say, to be called the <u>multiplicity</u> of the piece, of x-chains, each one running from the left-hand end to the right-hand end of the piece, and does not contain any y-line or point of odd valence except at its left-hand or right-hand end. Therefore, the pieces can be said to be the maximal parts of the 2-graph that are homogeneous in the x-direction.

From the way in which the pieces are constructed one sees immediately that all 2-graphs which are obtained by varying the lengths of the pieces at will and independently, contribute to χ'. Such a collection of 2-graphs that differ only in the length of one or more pieces will be called a <u>class</u> and symbolically represented by the same figure as the contributing 2-graphs, with the omission of all points except the point 1. Since all combinations of the lengths of the pieces are allowed, the contribution to $a_n(t_1)$ from a given class can be written as a product of factors, each piece giving one factor. The factor $c_\mu(t_1)$ that comes from a piece consisting of μ x-chains of which the length ℓ is varied is given by

$$c_\mu(t_1) = \sum_{\ell=1}^{\infty} (t_1^\mu)^\ell = \frac{t_1^\mu}{1-t_1^\mu} = \frac{t_1^\mu}{(1+t_1+t_1^2+\ldots+t_1^{\mu-1})(1-t_1)} \,. \tag{36}$$

It follows that the contribution to $a_n(t_1)$ from a given class of 2-graphs with n y-lines and r pieces of multiplicities $\mu_1, \mu_2, \ldots, \mu_r$ is

$$\prod_{i=1}^{r} c_{\mu_i}(t_1) = \frac{f(t_1)}{(1-t_1)^r} \,, \tag{37}$$

where $f(t_1)$ is regular and non-zero for $t_1 = 1$. Since $0 \leq r \leq n + 1$, the total coefficient $a_n(t_1)$ consists of terms $F_0(t_1)/(1-t_1)^{n+1}$, $F_1(t_1)/(1-t_1)^n$, ..., $F_{n+1}(t_1) \equiv 2$; each of the functions $F_0(t_1)$, $F_1(t_1)$, ... is the sum of a finite number of functions of the type $f(t_1)$, and is, therefore, regular for $t_1 = 1$. If we expand these functions in a Taylor series around $t_1 = 1$ and recollect terms with the same power of $1/(1-t_1)$, we can rewrite $a_n(t_1)$ as

$$a_n(t_1) = \left(\frac{1}{1-t_1}\right)^{n+1} \left[b_{n0} + (1-t_1)b_{n1} + (1-t_1)^2 b_{n2} + \dots \right], \tag{38}$$

where b_{n0}, b_{n1}, b_{n2}, ... are constants. Substituting (38) into (32) we find

$$\chi' - 1 = \frac{1}{1-t_1} \sum_{n=0}^{\infty} \left[b_{n0} + (1-t_1)b_{n1} + (1-t_1)^2 b_{n2} + \dots \right] \left(\frac{t_2}{1-t_1}\right)^n . \tag{39}$$

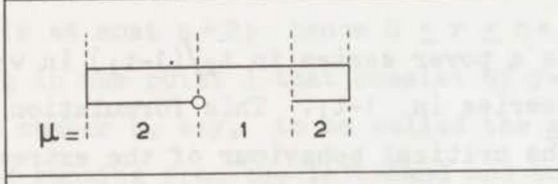
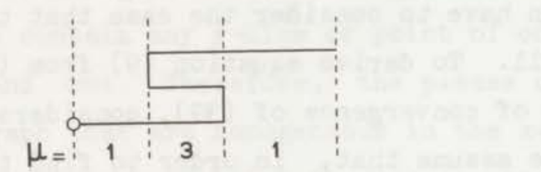
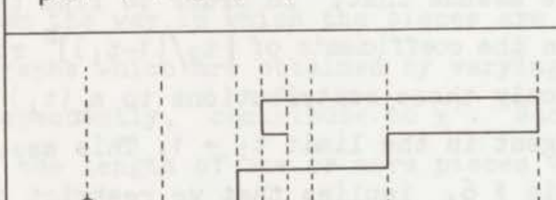
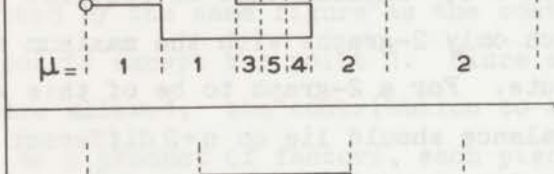
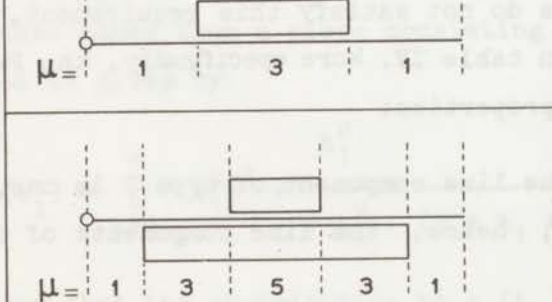
The right-hand side of equation (39) is a power series in $t_2/(1-t_1)$ in which the coefficients are themselves power series in $1-t_1$. This formulation is very useful for the investigation of the critical behaviour of the extremely anisotropic quadratic lattice: we then have to consider the case that t_1 is very close to unity and t_2 is very small. To derive equation (9) from (39) we must determine the limit of the radius of convergence of (39), considered as a series in $t_2/(1-t_1)$, for $1-t_1 \to 0$. We assume that, in order to find this limit, $1-t_1$ can be replaced by zero in the coefficients of $[t_2/(1-t_1)]^n$ right at the beginning, or, equivalently, that only those contributions to $a_n(t_1)$ have to be taken into account that are most divergent in the limit $t_1 \to 1$. This assumption, the nature of which will be discussed in § 6, implies that we restrict our attention to the constants b_{n0}, to which only 2-graphs with the maximum number of horizontal pieces ($r = n+1$) contribute. For a 2-graph to be of this kind its n y-lines and its two points of odd valence should lie on $n+2$ different infinite y-chains of the lattice. Many 2-graphs do not satisfy this requirement, e.g. the 2-graphs (a), (b), ..., (e) and (i) in table IV. More specifically, the 2-graphs in question should have the following properties:

a) the valence of both odd points in the line component of type 2 is one, the valence of the remaining points two; hence, the line components of type 2 are self-avoiding walks;
b) the valence of each point in a line component of type 0 is two, so that these line components are self-avoiding polygons.

The contribution to the coefficient b_{n0} from a given class of 2-graphs is easily calculated from (36) and (37). It is equal to f(1), i.e. to the product, taken over all pieces i=1, ..., n+1, of the number μ_i^{-1}, which is the value of the function $t_1^{\mu_i}/(1+t_1+\ldots+t_1^{\mu_i-1})$ for $t_1=1$. This contribution $\Pi_{i=1}^{n+1}\mu_i^{-1}$ will be called the weight of the class. A few examples of classes of 2-graphs, together with the splitting into pieces, the multiplicities μ of these pieces and the weight of the class, are given in table V.

TABLE V

Examples of classes of 2-graphs. The splitting into pieces is indicated by vertical dotted lines; μ is the multiplicity of the pieces.

class of 2-graphs	weight
	$1/2 \cdot 1/1 \cdot 1/2 = 1/4$
	$1/1 \cdot 1/3 \cdot 1/1 = 1/3$
	$1/1 \cdot 1/1 \cdot 1/3 \cdot 1/5 \cdot 1/4 \cdot 1/2 \cdot 1/2 = 1/240$
	$1/1 \cdot 1/3 \cdot 1/1 = 1/3$
	$1/1 \cdot 1/3 \cdot 1/5 \cdot 1/3 \cdot 1/1 = 1/45$

In practice a minor modification of the method of dividing a 2-graph into pieces is useful. First, we consider for a given class of 2-graphs all infinite horizontal lines (if any) that bisect exactly one y-line (see fig. 1). If we

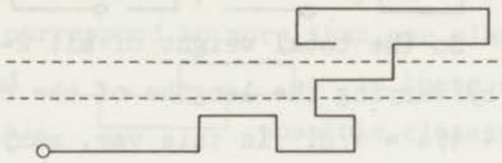
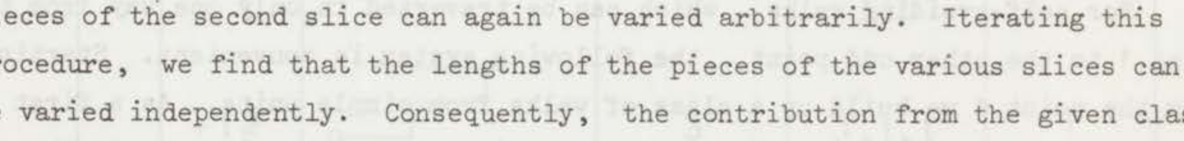


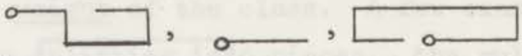
Fig. 1. Dividing the 2-graphs of a given class into slices.

delete the bisected y-lines, any 2-graph of the class falls into parts, to be called <u>slices</u>. Each slice contains exactly two odd points (and therefore is a 2-graph), each of which is an end of a deleted y-line or one of the odd points of the original 2-graph; the latter possibility exists only for the uppermost and the lowermost slice. To begin with, we now vary the lengths of the pieces of the slice, to which the point 1 belongs, arbitrarily. For any choice of these lengths, the position of the other odd point of the slice is uniquely determined; in turn, it determines the position of one of the odd points of the subsequent slice, which for this slice takes over the role of the point 1. Since the x-chains of this second slice can, by definition, not lie on the same height in the lattice as any x-chain of the first slice, the lengths of the pieces of the second slice can again be varied arbitrarily. Iterating this procedure, we find that the lengths of the pieces of the various slices can all be varied independently. Consequently, the contribution from the given class of 2-graphs is the product of the contributions from the classes of the separate slices.

The procedure of first dividing a 2-graph into slices reduces the calculation substantially. For instance, according to the partition into pieces one should treat separately the following classes of 2-graphs:

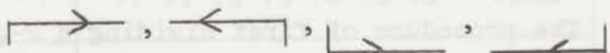
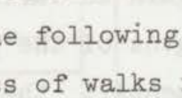


with respective weights 1/4, 1/12, 1/6, 1/6 and 1/3. By forming the slices first, however, one is left with three slices, each of which consists of a single x-chain. Hence, the total weight of all walks belonging to any of the above classes is $(1/1)^3 = 1$, which is also the sum of the weights of the separate classes, as it should.

To give another example, a similar simplification occurs for the third class of 2-graphs of table V, for which there exist two infinite horizontal lines that bisect exactly one y-line; they cut the 2-graphs into three slices of the structure  with respective factors 1/2, 1/1 and 1/4. So the total weight of all 2-graphs that can be built up from these slices by varying the lengths of the constituting pieces independently is $1/2 \cdot 1/1 \cdot 1/4 = 1/8$. In this way, many classes of 2-graphs are taken into account at once, of which the one given in table V, with weight 1/240, is only one. One sees that the partition into slices is in fact a generalization of the procedure that was used for the calculation of $a_1(t_1)$, the role of the x-chains being taken over by the slices, which include x-chains as a simple case.

§ 5. Calculation of the coefficients b_{n0}

Having established the way in which the contribution from a given class of 2-graphs is calculated, we have now to develop an efficient book-keeping system such that no 2-graph is omitted, or counted more than once. Usually such a system takes the form of a lexicographic ordering of some kind.

For self-avoiding walks, which can be traversed in only one way from the point 1 to the other odd point, the following system is convenient. Starting from the point 1 we build up a class of walks from simple units. As a first step we take an x-chain of arbitrary length directed to the right or to the left; we denote it by + or −, respectively. Then we proceed by successively adding units of the following kinds:  ; they are denoted by 1, 2, 3, 4, respectively. If each symbol representing a newly added unit is placed to the right of the previous symbol, a positive or negative integer is formed which will be used as a code for the class of walks considered. E.g., to the class  the code +21 is assigned. Because of symmetry it is sufficient to consider only those classes of walks whose code numbers start by +1 or +2; since the remaining classes can be obtained by a reflection with respect to the x-axis, or to the y-axis, or to both axes, their contribution can be taken into account by means of a factor 4. The classes with code numbers starting by +1 or +2 are ordered according to their code numbers. By running through all n-digit code numbers ($n \geq 1$) all classes of walks containing exactly n y-lines are taken into account; since we consider directed walks, no class is counted more than once.

The two classes containing no y-lines are represented by the codes + and -; evidently, there is a twofold rather than a fourfold symmetry in this case.

Two remarks should be made on this method of coding:

i) for some codes no walk can be realized, e.g. for all codes beginning by +141;

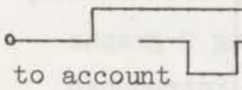
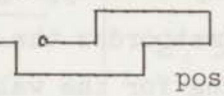
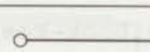
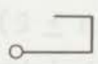
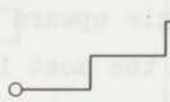
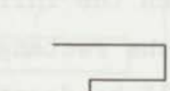
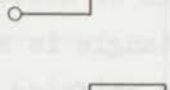
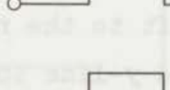
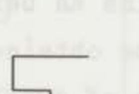
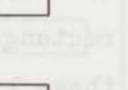
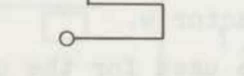
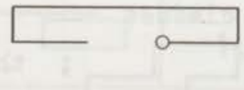
ii) some codes still correspond to more than one class, e.g. +1442 to  and  ; in these cases, care should be taken to account for all possible classes.

TABLE VI

The book-keeping for the walks containing up to two y-lines.

n	code	class of walks	number of slices	multiplicities	weight
0	+		1	1	1
1	+1		2	1,1	1
	+2		2	1,1	1
2	+11		3	1,1,1	1
	+12		3	1,1,1	1
	+13		1	1,1,1	1
	+14		1	1,1,2	1/2
	+21		3	1,1,1	1
	+22		3	1,1,1	1
	+23		1	2,1,2	1/4
	+24		1	1,1,2	1/2

For the classes of walks containing up to two y-lines ($n = 0, 1, 2$) the procedure is illustrated in table VI; we also indicate for each class the number of slices, the multiplicities μ_i of the pieces and the weight $\Pi_{i=1}^{n+1} \mu_i^{-1}$.

A similar book-keeping can be set up for each type of overlapping separately. Taking into account the fact that any overlapping is built up from exactly one walk and one or more polygons we first order the corresponding 2-graphs groupwise in conformity with the ordering for the walks that was introduced above. We consider only those cases in which the walk either is a single x-chain of the class with code + or starts by the unit +1 or the unit +2; the remaining, "reflected" overlappings can be taken into account by multiplying the contributions thus found by a factor 2 and 4, respectively. For the ordering of overlappings within a group, i.e. of overlappings containing the same walk, no general prescription will be given here. Instead, simple book-keeping rules will be used which are different for each type of overlapping; we give some examples for the 2-graphs with four or less y-lines.

a) Overlappings of the type 2-0 in which the graph 0 is a rectangle with two y-lines (which occur in the calculation of b_{n0} for $n \geq 2$) are generated as follows: first place the rectangle at an arbitrary height with respect to the walk in such a way that by shifting the rectangle upward or downward one can obtain an overlapping in which the rectangle and the most left-hand piece of the walk (and only this piece) overlap, and in which the infinite y-chain passing through the left-hand end of this piece cuts the rectangle. The number of overlappings which can be obtained in this way will be denoted by ν; obviously, $\nu \geq 2$. Next, the right-hand y-line of the rectangle is shifted to the right until one can obtain a new class of 2-graphs by shifting the rectangle vertically. This procedure is repeated until a further shift to the right does not yield a new class anymore. We then shift the left-hand y-line to the right and consider the different possibilities for the other y-line as before, and so on. At each stage, the number ν of overlappings that can be obtained by a vertical shift of the rectangle is at least two. Of the classes of overlappings of the walks + and +1 obtained in this way, the ones with the rectangle in lowest position are listed in table VII; the classes obtained from these by an upward shift of the rectangle are taken into account by the factor ν.

A generalization of this procedure is used for the case that the line component of type 0 belongs to one of the classes

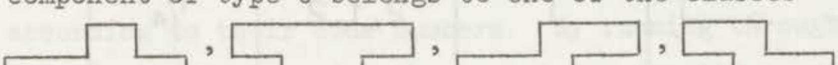
 ; these cases occur in the calculation of the coefficients b_{n0} for $n \geq 4$.

TABLE VII
The book-keeping for the overlappings of the walks + and +1 with a rectangle.

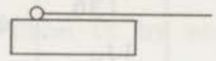
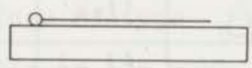
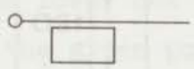
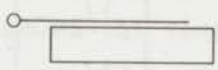
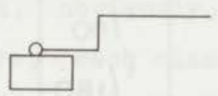
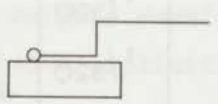
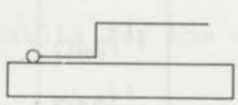
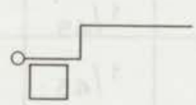
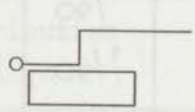
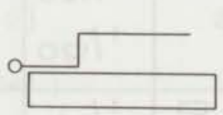
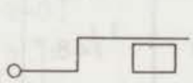
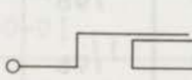
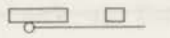
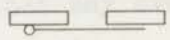
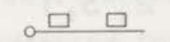
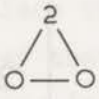
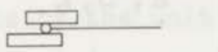
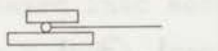
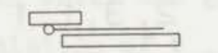
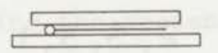
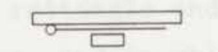
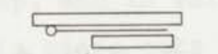
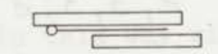
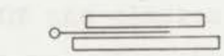
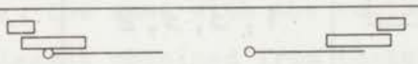
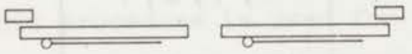
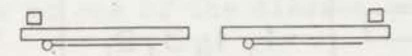
class of overlappings	multiplicities	weight	V
	2, 3, 1	$^1/_6$	2
	2, 3, 2	$^1/_{12}$	2
	1, 3, 1	$^1/_3$	2
	1, 3, 2	$^1/_6$	2
	2, 3, 1, 1	$^1/_6$	2
	2, 3, 3, 1	$^1/_{18}$	3
	2, 3, 3, 2	$^1/_{36}$	3
	1, 3, 1, 1	$^1/_3$	2
	1, 3, 3, 1	$^1/_9$	3
	1, 3, 3, 2	$^1/_{18}$	3
	1, 1, 3, 1	$^1/_3$	2
	1, 1, 3, 2	$^1/_6$	2

TABLE VIII
The book-keeping for the three-component 2-graphs contributing to b_{40}.

type of overlapping	class of overlappings	weight	V
O-2-O		$^1/_{18}$	4
		$^1/_{36}$	
		$^1/_{9}$	
		$^1/_{18}$	
		$^1/_{120}$	4
		$^1/_{120}$	
		$^1/_{240}$	
		$^1/_{240}$	
		$^1/_{90}$	
		$^1/_{90}$	
		$^1/_{180}$	
		$^1/_{320}$	
		$^1/_{320}$	
		$^1/_{180}$	
		$^1/_{240}$	
		$^1/_{240}$	
		$^1/_{45}$	
		$^1/_{45}$	
		$^1/_{90}$	
		$^1/_{90}$	
		$^1/_{120}$	
		$^1/_{120}$	
2-O-O		$^1/_{48}$	6
		$^1/_{48}$	
		$^1/_{96}$	
		$^1/_{96}$	

b) The three-component 2-graphs that contribute to b_{40} consist of an x-chain and two rectangles, both having two y-lines. The term $[2\text{-}0^2]$ in (30) does not contribute to b_{40} (or even to b_{41}), since the number of pieces is 3 rather than 5. The classes of overlappings which do contribute can be found from table VIII; each class of overlappings is understood to represent also the classes obtained from it by shifting the rectangles vertically without changing the type of overlapping; again, the number of classes represented is called ν. For the overlappings of type $\underset{0\text{-}0}{\overset{2}{\wedge}}$ the following book-keeping is used: fixing the upper rectangle in its most left-hand position with respect to the x-chain we first modify the lower one stepwise in the way described in (a) until all overlappings with the given position of the upper rectangle have been generated. Next, the upper rectangle is shifted to the next position and the procedure is repeated, etc. Some of the classes thus formed represent (in the above sense) the same overlappings; in such a case, only one of these representing classes is counted, with, obviously, $\nu = 4$; the other ones are left out from table VIII. Furthermore, each class representing overlappings of type $\underset{0\text{-}0}{\overset{2}{\wedge}}$ can be transformed into four classes of type 2-0-0 by suitable vertical shifts of the rectangles; the weights remain the same. Therefore, we have not included such classes of type 2-0-0 explicitly in table VIII, but only those which are not generated in this way.

The final results for the coefficients b_{00} up to b_{50} are shown in table IX.

TABLE IX

Type of 2-graphs	Contribution to χ'	b_{00}	b_{10}	b_{20}	b_{30}	b_{40}	b_{50}
2	[2]	2	8	25	77	226	662 1/6
2-0	-[2-0]			- 3	-21	-100 1/12	-414 11/12
0-2-0	[0-2-0]					2	$24\frac{3}{4}$
$\underset{0\text{-}0}{\overset{2}{\wedge}}$	$2.\left[\underset{0\text{-}0}{\overset{2}{\wedge}}\right]$					$2\frac{1}{2}$	20
2-0-0	[2-0-0]					$2\frac{3}{4}$	16
	Total amount	2	8	22	56	133 1/6	308

§ 6. Discussion

The main feature of the procedure that was developed in the previous two sections for the investigation of the reduced magnetic susceptibility χ' for an extremely anisotropic quadratic lattice is the replacement of the full expression (39) for $\chi'(t_1,t_2)$ by the approximate quantity $\chi_0'(t_1,t_2)$, given by

$$\chi_0'(t_1,t_2) \equiv \frac{1}{1-t_1} \sum_{n=0}^{\infty} b_{n0} \left(\frac{t_2}{1-t_1}\right)^n . \tag{40}$$

We now come to the discussion of the implications and the validity of this approximation. In the ferromagnetic region of the t_1, t_2-plane ($0 \leq t_1 \leq 1$, $0 \leq t_2 \leq 1$), to be denoted by F, the right-hand side of eq. (40) converges if $0 \leq t_2/(1-t_1) < b^{-1}$, where b^{-1} is the radius of convergence of the series $\Sigma_n b_{n0} x^n$. If we vary t_1 and t_2, $\chi_0'(t_1,t_2)$ will tend to infinity if either the factor $1/(1-t_1)$, or the power series, or both, tend to infinity. We consider only the first two cases.

a) If only the factor $1/(1-t_1)$ becomes infinite, $\chi_0'(t_1,t_2)$ diverges as $(1-t_1)^{-1}$, i.e. its critical behaviour is that of the susceptibility of the linear chain; the two-dimensional nature of the system does not manifest itself here. A sufficient condition for this to happen is that in F one approaches the point (1,0) along a straight line $b't_2 = 1-t_1$, with $b' > b$, i.e. a straight line lying between the positive t_1-axis and the straight line

$$bt_2 = 1 - t_1 . \tag{41}$$

According to the hypothesis made in section 4, (41) should be the tangent T in the point (1,0) to the line in the t_1,t_2-plane that represents the critical equation (7) (to be called the <u>critical line</u>); in other words, b should be 2. In order to check this value for b we shall compare it with the estimate for b that can be obtained from the coefficients b_{n0}, $n = 0, \ldots, 5$, calculated in the previous section. We use the ratio method [10]; it may be applied because of the positiveness of the coefficients b_{n0}. Therefore, we consider the behaviour of the ratios $\xi_n \equiv b_{n0}/b_{n-1,0}$ as a function of n, which, as is well known, should be of the form $b + b(p-1)/n$ for sufficiently large n, if, as we shall assume, the asymptotic behaviour of the power series $\Sigma_n b_{n0} x^n$ near its radius of convergence is of the well-known type:

$$\sum_{n=0}^{\infty} b_{n0} x^n \sim (1-bx)^{-p} \qquad (x \uparrow b^{-1}). \tag{42}$$

For the first few values of n, however, irregularities can occur. In particular, for "loosely-packed" lattices, such as the quadratic and the simple cubic lattice, oscillations between even and odd values of n take place. Since in fact only a finite number of coefficients b_{n0} are available, b and p can be determined only with a limited accuracy.

We have used a least-squares method in order to find the values of b and p for which the form $b + b(p-1)/n$ lies closest to the available ratios ξ_n, $n=1, \ldots, 5$. Since the above-mentioned irregularities in ξ_n become less important for larger n, it is reasonable to attach, in one way or another, more significance to the ratios for larger n than to the ratios for smaller n. One can achieve this, e.g., by leaving the first few ξ_n completely out of consideration. Alternatively, one can count the ξ_n with certain positive weight factors which increase with n; the ratio of the weight factors for two subsequent values of n should approach unity for $n \to \infty$, since for large values of n all ratios become equally significant. We have followed both procedures, separately and in combination, choosing the simplest weight factor satisfying the above conditions, viz. n itself. The accuracy of the calculation can be estimated by carrying it out for different sets of subsequent ξ_n. From the set of values for b and p obtained in this way we derive the following estimates for b and p:

$$b = 2.02 \pm 0.03$$
$$p = 1.77 \pm 0.03. \tag{43}$$

The value found for p suggests strongly that p is equal to the critical index γ of the susceptibility of the isotropic quadratic lattice, which is exactly 7/4. If this is true, we may start from the assumption that the series $\Sigma_n b_{n0} x^n$ diverges for $x \uparrow b^{-1}$ as $(1-bx)^{-7/4}$, which in turn will give a somewhat more reliable estimate of b [10]; the result is

$$b = 2.01 \pm 0.03. \tag{44}$$

In view of the fact that the value 2 for b falls indeed within the limits of accuracy of (44) we may, conversely, use b = 2 rather than p = 7/4 as an extra piece of information. In that case we find

$$p = 1.77 \pm 0.02. \tag{45}$$

By the preceding analysis we feel justified in concluding that the series $\Sigma_n b_{n0} x^n$ diverges for $x = \frac{1}{2}$, so that in order to find the radius of convergence of the series (39) in the limit $1-t_1 \to 0$ it is indeed allowed to put t_1 equal to 1 in its coefficients right at the beginning. In other words, the asymptotic form of the critical equation (7) for $t_1 \to 1$ can be found from the numbers b_{n0} alone.

b) Let us now return to equation (40) and consider the case that the power series in $t_2/(1-t_1)$ diverges, while the factor $1/(1-t_1)$ remains finite. This will take place in the limit where (t_1,t_2) approaches a point $(\tau, b^{-1}(1-\tau)) \neq (1,0)$ of the tangent T. It follows from the foregoing analysis that under this condition the critical behaviour of χ_0' is of the same nature as that of the susceptibility of the isotropic quadratic lattice; we call this a two-dimensional critical behaviour. This divergence of χ_0' on the tangent T demonstrates most clearly the shortcoming of the approximate expression (40) for the susceptibility χ'. For, in contrast to χ_0', χ' should be finite on T; its divergence should not occur before the critical line itself is reached. Obviously, the sum of all remaining series $(1-t_1)^{m-1}\Sigma_n b_{nm} [t_2/(1-t_1)]^n$, $m = 1, 2, \ldots$ in equation (39), which also contribute to χ' and which we have so far neglected, has to diverge as the tangent T is approached in order that the divergence of χ_0' is compensated for. It is natural to conjecture that each of the series separately diverges as T is approached, but a definite conclusion can be reached only by an explicit calculation of the numbers b_{nm} for $m \geq 1$, which, even for $m = 1$, is considerably more complicated than the calculation of the numbers b_{n0}. It is satisfying, however, that the critical exponent p of χ_0' is, within the limits of accuracy, equal to the critical exponent $\gamma = 7/4$ of the susceptibility of the isotropic quadratic lattice. This is in agreement with the conjecture made in the introduction, which implies that, for sufficiently general values of t_1 and t_2, the susceptibility diverges as a function of Δ_2 alone.

Summarizing we can say that the divergence caused by the factor $1/(1-t_1)$ in the equations (39) and (40) is the same, so that in this respect the approximation for χ' is correct. In contrast to this, the divergence caused by the power series in (40) predicts a too small domain of convergence of χ'; one may expect that the discrepancy which exists in this respect between the equations (39) and (40) decreases if one considers values of t_1 that lie closer to 1.

Throughout this paper we have considered ferromagnetic Ising systems on an extremely anisotropic quadratic lattice. One easily convinces oneself, however, that in the procedure for the calculation of the susceptibility introduced in this paper no use is made of the sign of the weak coupling constant J_2.

On the other hand, the restriction to positive values of the strong coupling constant J_1 is essential, because an extensive use is made of the fact that t_1 is close to 1. Hence our method applies equally well to the "semi-anti-ferromagnetic" case $J_1 > 0$, $J_2 < 0$, i.e. the series $\Sigma_n b_{n0} x^n$ describes the asymptotic behaviour of the susceptibility, not only in the limit $t_1 \uparrow 1$, $t_2 \downarrow 0$, but also in the limit $t_1 \uparrow 1$, $t_2 \uparrow 0$. A closer investigation of the series yields indeed information on the semi-antiferromagnetic case. We shall consider this case in some detail, together with other antiferromagnetic cases, in a separate paper.

Having verified that the procedure introduced in this paper leads to the correct asymptotic form of the critical equation of the anisotropic quadratic lattice we shall in a second paper apply it to the simple cubic lattice of which two of the three coupling constants are very small compared to the third one.

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CRITICAL BEHAVIOUR OF ISING SYSTEMS ON EXTREMELY ANISOTROPIC LATTICES

II. The simple cubic lattice

Synopsis

The critical behaviour of the magnetic susceptibility χ of a spin-$\frac{1}{2}$ Ising system on a simple cubic lattice in which the coupling constants along two of the three lattice axes, J_2 and J_3, are very small in comparison with the coupling constant along the remaining lattice axis, J_1, is investigated on the basis of the series expansion of χ in the variables $t_r = \tanh\beta J_r$, $r = 1, 2, 3$. For $t_2, t_3 \ll 1$, $1-t_1 \ll 1$ the reduced susceptibility $\chi' \equiv (kT/\mu^2)\chi$ is found to behave as

$$\chi'(t_1,t_2,t_3) \approx \chi'_0(t_1,t_2,t_3) \equiv \frac{1}{1-t_1} \sum_{m,n=0}^{\infty} b_{mn0} \left(\frac{t_2}{1-t_1}\right)^m \left(\frac{t_3}{1-t_1}\right)^n ,$$

where the b_{mn0} are constants. The three-dimensional nature of the system is contained in the double power series $\Sigma_{m,n} b_{mn0} [t_2/(1-t_1)]^m [t_3/(1-t_1)]^n$, of which the singular behaviour is investigated for the cases $t_2 = at_3$, $a = 1, 2, 4, 6, 8$. Rewriting for these cases the double power series as a single power series $\Sigma_n b_{n0}(a) [t_2/(1-t_1)]^n$ and assuming for this series a singular behaviour of the well-known type $[1-b(a)t_2/(1-t_1)]^{-p(a)}$ we find in all five cases that the power $p(a)$ is consistent with the value $\gamma = \frac{5}{4}$ of the critical exponent of χ for the isotropic simple cubic lattice. Using this value we find for $b(a)$ the values 6.10 ± 0.02, $4.48^5 \pm 0.02$, 3.54 ± 0.02, 3.19 ± 0.02, 2.99 ± 0.02 for $a = 1, 2, 4, 6, 8$ respectively. These numerical results are used for the investigation of the surface which represents the critical equation in the t_1, t_2, t_3-space. It turns out that very close to the point (1,0,0) this surface behaves as a cone with its apex in (1,0,0); in this respect the Bethe-Peierls approximation according to which the surface behaves near (1,0,0) as a plane rather than as a cone, is essentially in error.

§ 1. Introduction

This paper, which is the second one in a series of papers on the critical behaviour of spin-$\frac{1}{2}$ Ising systems on extremely anisotropic lattices, is devoted to a study of the simple cubic (s.c.) lattice in which the coupling constants along two of the three lattice axes, J_2 and J_3, are very small in comparison with the coupling constant along the remaining lattice axis, J_1. To this end we generalize the technique that was introduced in the first paper [1] (to be referred to as I) for the investigation of the extremely anisotropic quadratic lattice. We write the initial magnetic susceptibility χ as a triple power series in the variables $t_r = \tanh\beta J_r$, $r = 1, 2, 3$, and rearrange it into a double power series in t_2 and t_3 with coefficients depending on t_1. Analogously to the situation for the quadratic lattice, these coefficients are sums of functions of t_1 which diverge for $t_1 \to 1$. Our main interest concerns the asymptotic form of the critical equation

$$\Delta_3(t_1,t_2,t_3) = 0 \tag{1}$$

of the s.c. lattice, which equation is not known exactly, in contrast to that of the quadratic lattice. We shall suppose that, in order to find this asymptotic form, it is sufficient to consider only those contributions to the above-mentioned coefficients that show the strongest divergence for $t_1 \to 1$; in I, this hypothesis was found to give reliable results. The procedure of calculating the most divergent contributions can be taken from I with only a simple straightforward extension. Therefore, the terminology of I is used throughout this paper. The extended procedure, together with the actual calculation of the first few coefficients, is given in § 2.

In § 3 we investigate the asymptotic behaviour of χ, using the coefficients found in § 2. We restrict ourselves to a few special cases, viz. those characterized by $t_2 = at_3$, where $a = 1, 2, 4, 6, 8$. For each of these cases the asymptotic behaviour of χ is described by a power series in one variable, $t_2/(1-t_1)$, which is analysed by means of the ratio method. The results seem to confirm another hypothesis, made in I, viz. that the critical behaviour of χ can be described in terms of the function Δ_3 alone; furthermore, they strongly suggest that the behaviour of Δ_3 itself near the point $(t_1,t_2,t_3) = (1,0,0)$ is essentially different from that predicted by the Bethe-Peierls approximation.

§ 2. The magnetic susceptibility of the extremely anisotropic simple cubic lattice

In order to obtain the high-temperature series expansion of the susceptibility χ of an Ising system on an extremely anisotropic s.c. lattice for which the coupling constants J_2 and J_3 are much smaller than J_1 we use the procedure developed in I. The reduced susceptibility $\chi' \equiv (kT/\mu^2)\chi$ is written as a triple power series in t_1, t_2 and t_3:

$$\chi' = 1 + \sum_{\substack{n_1,n_2,n_3=0 \\ n_1+n_2+n_3 \geq 1}}^{\infty} a_{n_1n_2n_3} t_1^{n_1} t_2^{n_2} t_3^{n_3} \; . \tag{2}$$

Since we consider Ising systems with $t_2, t_3 << t_1$, we rearrange the series into a double series in t_2 and t_3 with coefficients depending on t_1; this is achieved by first carrying out the summation over n_1:

$$\left.\begin{aligned} &\chi' = 1 + \sum_{m,n=0}^{\infty} a_{mn}(t_1) t_2^m t_3^n \\ &\text{with} \\ &a_{mn}(t_1) = \sum_{\substack{n_1=0 \\ n_1+m+n \geq 1}}^{\infty} a_{n_1mn} t_1^{n_1} \end{aligned}\right\} \tag{3}$$

In equation (3) the term $a_{mn}(t_1)t_2^m t_3^n$ is the contribution to χ' from 2-graphs containing fixed numbers m and n of y- and z-lines, respectively, and an arbitrary number of x-lines. The contribution to $a_{mn}(t_1)$ from a given class of 2-graphs containing m y-lines and n z-lines is calculated as follows.

First, we divide each 2-graph into pieces by drawing all infinite planes which go through the y- or z-lines of the 2-graph or through its points of odd valence, and which are perpendicular to the x-axis. Performing this procedure for each class of 2-graphs we find that the analytic structure of $a_{mn}(t_1)$ is the same as that of the coefficients $a_n(t_1)$ given by equation (38) of I: it is a sum of terms $F_0(t_1)/(1-t_1)^{m+n+1}$, $F_1(t_1)/(1-t_1)^{m+n}$, ..., $F_{m+n+1}(t_1)$, which are the contributions from 2-graphs containing m+n+1, m+n, ..., 0 pieces. Each of the functions $F_0(t_1)$, $F_1(t_1)$, ... is the sum of a finite number of functions that are regular (and non-zero) for $t_1 = 1$, and is therefore itself regular for $t_1 = 1$; the function $F_{m+n+1}(t_1)$ is a constant, equal to the coefficient a_{mn} occurring in the double power series for the reduced susceptibility of the quadratic lattice. Hence, equation (3) can be rewritten as follows:

$$\chi' - 1 = \frac{1}{1-t_1} \sum_{m,n=0}^{\infty} \left\{ b_{mn0} + (1-t_1) b_{mn1} + (1-t_1)^2 b_{mn2} + \dots \right\} \frac{t_2^m t_3^n}{(1-t_1)^{m+n}} , \qquad (4)$$

in which b_{mn0}, b_{mn1}, etc. are constants.

For the investigation of the asymptotic form of the critical equation of the s.c. lattice for t_2 and t_3 going simultaneously to zero we assume that this asymptotic form is determined by the coefficients b_{mn0} alone. For the quadratic lattice the corresponding assumption has been shown to be correct within the limits of accuracy in I. In the 2-graphs contributing to b_{mn0} the number r of pieces is maximal ($r = m + n + 1$), line components of type 2 are self-avoiding walks, and line components of type 0 are self-avoiding polygons.

The weight of a given class of 2-graphs can be found in the same way as in I. Here, too, a simplification of the calculations is possible. It is achieved by generalizing the division of a 2-graph into slices, introduced in I, in the following way. We first form the orthogonal projection of the 2-graph under consideration onto the yz-plane, representing each infinite x-chain on which there are points of the 2-graph by a point, and each line in the 2-graph connecting two points on different infinite x-chains by a line connecting the points representing these chains. Next, we consider all articulation lines of the multigraph thus obtained, i.e. each line which has the property that its removal disconnects the multigraph. We then delete the y- or z-lines in the original 2-graph of which these articulation lines are the projections. In this way the 2-graph is split into one or more disconnected parts. All classes of 2-graphs which can be obtained by varying the lengths of the pieces of these <u>parts</u> independently of each other contribute to χ'. For, each choice of the lengths of the pieces of a given part fixes the position of one and only one "point of entrance" in each of the other parts to which it was connected by a y- or a z-line before the deletion, i.e. the end point of this y- or z-line in that part. By this fact, in combination with the property that by definition no x-chain of a part can be collinear with any x-chain of another part, the lengths of the pieces of each part can be chosen independently of the lengths of the pieces of any other part. Hence, all 2-graphs thus obtained can be taken into account at once by forming the product of the weights of all parts. If we perform the analogous procedure for the 2-graphs on the quadratic lattice, the role of the yz-plane being taken over by the y-axis, the parts of the 2-graphs thus formed are identical with the slices.

The book-keeping used to take account of all classes is an extension of that introduced in I for the quadratic lattice. In the following we consider the problem of obtaining all 2-graphs with a fixed total number of y- and z-lines rather than those with fixed numbers of y- and z-lines separately. We begin with the self-avoiding walks for which the simplest extension of the book-keeping goes as follows.

For convenience we introduce the concept of a +x step (chain) in the description of a walk. By this we shall understand an x-line (chain) that is traversed in the +x direction if one follows the walk, starting from the point 1; analogously we define +y, +z, -x, -y, -z steps (chains).

We shall build up a class of walks from 2 + 8 rather than 2 + 4 different types of units. The units are represented by the same code symbols +, -, 1, 2, 3, 4 as used in I, but each of the symbols 1, 2, 3, 4 has now a double meaning: the symbol 1 is used not only for a unit consisting of a +y step followed by a +x chain, but also for a unit consisting of a +z step followed by a +x chain; the use of the symbols 2, 3, 4 is analogously extended.

By running through all n-digit code numbers, including the ones for which on the quadratic lattice no walk can be realized, each class of walks in which the total number of y- and z-lines is n is obtained. In contrast to the situation for walks on the quadratic lattice (to be called Q-walks henceforth) there may be realized for each code number at least one walk on the cubic lattice, viz. the walk obtained by taking for each symbol 1 or 2 occurring in the code number the corresponding unit containing a +y step and for each symbol 3 or 4 the corresponding unit containing a -z step. In such a walk no two x-chains are collinear because in order to go from one of these chains to the other one one has to traverse, apart from x-chains, +y steps and -z steps only. Since for two x-chains to be collinear a necessary (and sufficient) condition is that the numbers of +y and -y steps between these x-chains should be equal (and similarly for the z steps) no two x-chains in the walk considered are collinear.

This book-keeping is equivalent to the multi-valued mapping of Q-walks onto walks on the s.c. lattice (C-walks) defined by the following rules:

i) each +x step (-x step) remains a +x step (-x step);

ii) each +y step (-y step) remains a +y step (-y step) or becomes a +z step (-z step). From now on we shall say that a class of C-walks belonging to a given code number corresponds to the class of Q-walks (if it exists) belonging to the same code number and vice versa; the p^{th} x-chain in a C-walk (the walk being traversed starting from the point 1) is said to correspond to the

p^{th} x-chain in the corresponding Q-walk.

In principle, the weights of the various classes of C-walks belonging to a given code number may be different; however, they are never smaller than the weight of the corresponding class of Q-walks.

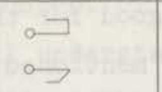
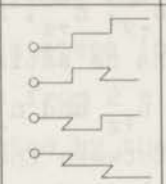
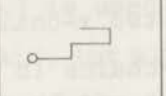
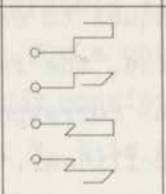
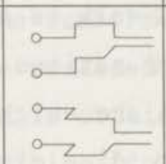
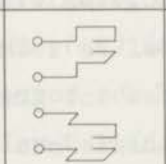
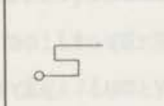
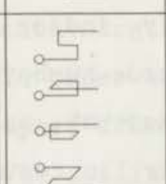
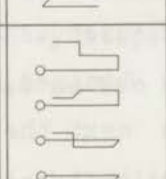
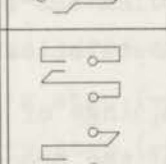
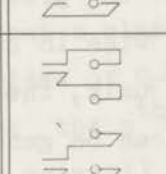
To prove this, we first observe that, if two x-chains in a Q-walk are not collinear, then the same holds good for the corresponding x-chains of all corresponding C-walks. For, as mentioned above, if two x-chains in a C-walk are collinear the numbers n_{+y}, n_{-y}, n_{+z}, n_{-z} of +y, -y, +z, -z steps, respectively, between these x-chains satisfy the relations $n_{+y} = n_{-y}$; $n_{+z} = n_{-z}$. In that case, the numbers $n_{+y} + n_{+z}$ and $n_{-y} + n_{-z}$, which are the numbers of +y and -y steps, respectively, between the corresponding x-chains in the corresponding Q-walk, are also equal to each other; so the latter x-chains are also collinear. On the other hand, the fact that two x-chains in a Q-walk are collinear does not imply that the corresponding x-chains in each of its corresponding C-walks are also collinear. So each combination of lengths of x-chains which is allowed in a Q-walk is also allowed in the corresponding C-walks, but in addition new combinations of lengths may occur in the latter walks. As a direct consequence, the weight of a class of C-walks is never smaller than that of the corresponding class of Q-walks. For those classes of Q-walks which have a weight equal to 1, the maximal possible value for a weight, this implies that the weights of the corresponding classes of C-walks are also 1, so that the piece-structure of these C-walks need not be investigated in detail.

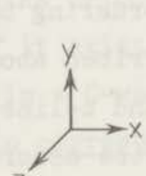
The walks with one and two y- and/or z-lines are listed in table I; we consider only the walks with a code number starting by +1 or +2, since the remaining ones can be taken into account, as in I, by multiplying the weights by a factor 4. For each code number the resulting C-walks are listed in "lexicographical" order: first the walk which is identical to the corresponding Q-walk, then the walk which can be obtained from it by changing the last y-line, and only this one, into a z-line, next the walk, which can be obtained from the first walk by changing the second last y-line, and only this one, into a z-line, in the fourth place the walk with both last y-lines changed into z-lines.

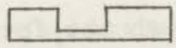
The book-keeping for overlappings of the type 2-0 can be set up by a straightforward generalization of the procedure developed for the quadratic lattice, the ordering being groupwise in conformity with the book-keeping for the walks described above. Obviously, the polygons to be considered may contain both y- and z-lines; they can be generated from the polygons on the quadratic lattice according to the rules (i) and (ii), provided we consider

TABLE I

The book-keeping for the C-walks with code numbers starting by + 1 and + 2. For comparison also the corresponding Q-walks are listed. For the interpretation of the diagrams for C-walks a tripod formed by the positive x-, y- and z-axes is added.

code	Q-walk	weight	C-walks	weight
+ 1		1		1 1
+ 2		1		1 1
+11		1		1 1 1 1
+12		1		1 1 1 1
+13		1		1 1 1 1
+14		1/2		1/2 1 1 1/2
+ 21		1		1 1 1 1
+ 22		1		1 1 1 1
+ 23		1/4		1/4 1 1 1/4
+ 24		1/2		1/2 1 1 1/2



"forbidden" classes like  as well; the role of the starting point in (i) and (ii) can be taken over by any point of the polygon.

As a final example we consider the 2-graphs of the types 0-2-0, $\overset{2}{\diagup\diagdown}_{0-0}$ and 2-0-0, in which the total number of y- and z-lines is four. They can be found in a simple way with the aid of the diagrams of table VIII of I; allowing for the fact that both rectangles can be in the xy- or in the xz-plane, we find that

a) each diagram for the types 0-2-0 and $\overset{2}{\diagup\diagdown}_{0-0}$ represents 16 instead of 4 2-graphs;

b) those diagrams for the type 2-0-0 that have a corresponding diagram for the type $\overset{2}{\diagup\diagdown}_{0-0}$ represent 24 instead of 4 2-graphs, the remaining ones 28 instead of 6 2-graphs.

Using the procedure described above the coefficients b_{mn0} with $0 \leq m, n \leq 5$, $0 \leq m+n \leq 5$, have been evaluated. The results are listed in table II, in which for symmetry reasons only the b_{mn0} with $m \geq n$ are given.*) For the interpretation of the first column see I, table IX.

TABLE II

	$b_{00\,0}$	$b_{10\,0}$	$b_{20\,0}$	$b_{11\,0}$	$b_{30\,0}$	$b_{21\,0}$	$b_{40\,0}$	$b_{31\,0}$	$b_{22\,0}$	$b_{50\,0}$	$b_{41\,0}$	$b_{32\,0}$
[2]	2	8	25	64	77	328	226	1416	2329	$662\frac{1}{6}$	5522	$13269\frac{7}{9}$
-[2-0]	-	-	- 3	-	-21	- 24	$-100\frac{1}{12}$	- 264	$- 282\frac{1}{3}$	$-414\frac{11}{12}$	$-1772\frac{2}{3}$	-2704
[0-2-0]	-	-	-	-	-	-	2	-	4	$24\frac{3}{4}$	34	57
$2\,[\overset{2}{\diagup\diagdown}_{0-0}]$	-	-	-	-	-	-	$2\frac{1}{2}$	-	5	20	20	42
[2-0-0]	-	-	-	-	-	-	$2\frac{3}{4}$	-	9	16	22	61
Total amount	2	3	22	64	56	304	$133\frac{1}{6}$	1152	$2064\frac{2}{3}$	308	$3825\frac{1}{3}$	$10725\frac{7}{9}$

§ 3. Discussion

We now come to the investigation of the quantity χ_0', given by

$$\chi_0'(t_1,t_2,t_3) \equiv \frac{1}{1-t_1} \sum_{m,n=0}^{\infty} b_{mn0} \left(\frac{t_2}{1-t_1}\right)^m \left(\frac{t_3}{1-t_1}\right)^n , \tag{5}$$

by which we have approximated the reduced susceptibility of the extremely anisotropic s.c. lattice. Analogously to the corresponding quantity for the

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*) See, however, the note at the end of this chapter.

quadratic lattice, χ_0' can diverge both through the factor $1/(1-t_1)$ and through the double power series. We shall consider only the latter case since the discussion of the divergence through the factor $1/(1-t_1)$ can be taken unchanged from the last section of I.

In order to see in which part of the "ferromagnetic" region $0 \leq t_r \leq 1$ $(r = 1, 2, 3)$ of the t_1, t_2, t_3-space the double power series in $t_2/(1-t_1)$ and $t_3/(1-t_1)$ converges, we first consider an arbitrary plane $t_2 = at_3$ with $a > 0$ through the t_1-axis. In this plane the double power series reduces, through the substitution $t_3 = t_2/a$, to a single power series in $t_2/(1-t_1)$, to be denoted by $\Sigma_{n=0}^{\infty} b_{n0}(a)[t_2/(1-t_1)]^n$ henceforth. That part of the ferromagnetic region of the plane in which the latter series converges is bounded by a straight line of the form $t_2/(1-t_1) = b^{-1}(a)$, where $b^{-1}(a)$ is the radius of convergence of the series $\Sigma_n b_{n0}(a)x^n$. Letting a vary from 0 to ∞ we find that the part of the ferromagnetic region in the t_1, t_2, t_3-space in which the double power series converges is bounded by a ruled surface which is part of a cone with its apex in the point (1,0,0); this cone, to be called the critical cone in (1,0,0), may degenerate into a plane through (1,0,0).

In order to investigate the cone we shall determine some of the straight lines lying on it, or, equivalently, calculate the number $b(a)$ for some values of a, viz. a = 1, 2, 4, 6, 8. As in I, § 6 we assume that the singular behaviour of the series $\Sigma_n b_{n0}(a)x^n$ is of the type:

$$\sum_{n=0}^{\infty} b_{n0}(a)x^n \sim [1 - b(a)x]^{-p(a)}, \quad x\uparrow b^{-1}(a), \tag{6}$$

which implies that the successive ratios $\xi_n(a) \equiv b_{n0}(a)/b_{n-1\,0}(a)$ behave as $b(a) + b(a)\left\{p(a)-1\right\}/n$ for large n. The right-hand member of (6) describes the critical behaviour of χ_0' and hence of the susceptibility of the extremely anisotropic s.c. lattice, if (t_1, t_2, t_3) approaches in the plane $t_2 = at_3$ a point $\neq (1,0,0)$ of the critical cone; the equation

$$b(a)\,\frac{t_2}{1-t_1} = 1 \tag{7}$$

should, for any fixed a, be the asymptotic form of the critical equation

$$\Delta_3(t_1, t_2, t_2/a) = 0. \tag{8}$$

The procedure used for the analysis of the power series $\Sigma_n b_{n0}(a)x^n$ is the same as the one introduced in I, § 6; however, the situation encountered here differs in two points from the situation in I. First, we have no exact expression for $b(a)$ at our disposal with which to compare our results and by means of which $p(a)$ could be determined more accurately. Secondly, the power $p(a)$ may in principle depend on a, so that different values may be found for the five cases considered here. However, on account of the hypothesis introduced in I, § 1, that the critical behaviour of thermodynamic quantities can be described in terms of Δ_d alone, we do not expect such a dependence; more specifically, this hypothesis implies that $p(a)$ is equal to the critical index γ of the susceptibility of the isotropic s.c. lattice, which is believed to be 5/4, although this value has not been established rigorously. Accordingly, we have used this value for $p(a)$ in order to determine $b(a)$ more accurately.

The results of the analysis are listed in table III.

TABLE III

The parameters $b(a)$ and $p(a)$ as estimated from the coefficients b_{mn0}; the values of $b(a)$ in the last column are found if $p(a)$ is put equal to 5/4.

a	b(a)	p(a)	b(a)
1	6.08 ± 0.10	1.26 ± 0.05	6.10 ± 0.02
2	4.47 ± 0.08	1.26 ± 0.06	4.48^5 ± 0.02
4	3.53 ± 0.05	1.27 ± 0.06	3.54 ± 0.02
6	3.14 ± 0.05	1.30 ± 0.06	3.19 ± 0.02
8	2.94 ± 0.05	1.31 ± 0.06	2.99 ± 0.02

Note that the quadratic lattice corresponds to $a = \infty$, so that one can write formally $b(\infty) = 2$, $p(\infty) = 7/4$.

It is a striking feature that the values for $p(a)$ given in the second column of table III do indicate a dependence on a. However, we believe strongly that this dependence is only a seeming one. A possible explanation can be found from the fact that for all five values of a we have used the <u>same</u> finite number of coefficients $b_{n0}(a)$. Our procedure for determining $b(a)$ and $p(a)$ is equivalent to drawing that straight line ℓ in the $1/n$, ξ_n-plane, that fits best to the points $(1/n, \xi_n(a))$, $n = 1, \ldots, 5$. The intersection of ℓ with the ξ_n-axis should take place in the point $(0, b(a))$, the intersection with the 1/n-axis in the point $\left(-(p(a)-1)^{-1}, 0\right)$. For large values of a, however, the coefficients $b_{n0}(a)$ will tend to their limits b_{n0} which are the coefficients for the quantity χ_0' for the quadratic lattice; hence the points $(1/n, \xi_n(a))$ will tend to the corresponding points for the quadratic lattice.

In order to find the true asymptotic behaviour of the $\xi_n(a)$ for $n \to \infty$, one would have to take into account more and more coefficients $b_{n0}(a)$ as a increases. If, in accordance with our hypothesis, the complete series diverges for all a with a power 5/4, then for each value of a one would find that the straight line ℓ towards which the points $(1/n, \xi_n(a))$ seem to tend asymptotically intersects the 1/n-axis in a point which comes closer to the point $(-(5/4-1)^{-1}, 0) = (-4,0)$ as more points $(1/n, \xi_n(a))$ are calculated and taken into account; more specifically, the intersection point can be written as $(-(p-1)^{-1}, 0)$, where p approaches the value 5/4 from above, so that the lines (which deviate less and less from the line which would be obtained if <u>all</u> points $(1/n, \xi_n(a))$ could be taken into account) become less steep. As a consequence, the intersection point with the ξ_n-axis approaches the point $(b(a), 0)$ from below; hence, the value for b(a) as estimated from a finite number of terms in the series will be too small.

On the other hand, one may use the correct value for p(a) as an extra piece of information right from the beginning, so that the line ℓ' which is fitted to the points $(1/n, \xi_n(a))$ is of the form

$$\xi_n(a) = b(a)\left(1 + \frac{1}{4n}\right). \tag{9}$$

Thus, ℓ' is "forced" to go through (-4,0) <u>exactly</u>; consequently, its intersection point with the ξ_n-axis will lie <u>above</u> the point $(b(a), 0)$, so that the value of b(a) thus determined will be too large.

Obviously, the values found for b(a) and p(a) will deviate more from the exact values as a increases. Therefore, the information obtained from the coefficients $b_{n0}(a)$, $n = 0, \ldots, 5$, will become less reliable and care should be taken even for the cases a = 6, 8 as can be seen from table III. From the foregoing it follows that the exact value for b(a) is somewhere between the number given in the first and that given in the third column of that table. Note that the former number is smaller than the latter one as it should be. Taking into account the remarks made above about the line ℓ we conclude that all five values for p(a) listed in table III are consistent with the value 5/4.

Returning to the critical cone itself, we find an important feature from the numerical results for b(a), viz. that this cone is not degenerated into a plane through (1, 0, 0). For, such a plane would, on account of symmetry, be described by an equation of the form

$$1 - t_1 - ct_2 - ct_3 = 0. \tag{10}$$

The constant c could be calculated directly by observing that (10) should reduce to the asymptotic form of the critical line in the t_1, t_2- (or t_1, t_3-) plane if t_2 (or t_3) is put equal to zero; it is found that

$$c = 2. \tag{11}$$

The intersection of the plane defined by the equations (10) and (11) with the plane $t_2 = at_3$, where a is an arbitrary constant, is given by the equation

$$1 - t_1 = 2(1 + \frac{1}{a})t_2. \tag{12}$$

On the other hand, the intersection with the plane $t_2 = at_3$ should be given by equation (7), so that the numbers b(a) and $2(1 + \frac{1}{a})$ should be equal for all a. A comparison between the values of $2(1 + \frac{1}{a})$ for a = 1, 2, 4, 6, 8 and the estimated values for b(a) listed in table III shows a manifest discrepancy between these two sets of numbers which exceeds by far the limits of accuracy of b(a). Hence, the critical cone is indeed a (non-trivial) cone and not a plane. Equivalently, one can say that the surface in the t_1, t_2, t_3-space representing (1) (which on the analogy of the critical line for the quadratic lattice may be called the <u>critical surface</u>) behaves near (1, 0, 0) as a cone with its apex in this point, and not as a plane. This behaviour is in marked contrast to that of the critical surface predicted by the Bethe-Peierls approximation. The equation for the Curie-temperature in this approximation can be derived by extending the procedure followed in I, § 2a to the s.c. lattice in a straightforward way, which yields

$$2 - \sum_{r=1}^{3} \exp(-2\beta_c J_r) = 0. \tag{13}$$

For J_2, J_3 << J_1 this equation reduces to

$$1 - t_1 \approx 4(t_2 + t_3) \qquad (t_2, t_3 \to 0), \tag{14}$$

which expresses the fact that the critical surface implied by the Bethe-Peierls approximation <u>does</u> behave as a plane near (1, 0, 0). In this respect, the Bethe-Peierls approximation is essentially in error, whereas for the quadratic lattice it led to the right type of critical equation, viz.

$1 - t_1 - t_2 - Ct_1t_2 = 0$, but with the coefficient $C = 3$ rather than $C = 1$.

The conical behaviour near (1, 0, 0) of the exact critical surface can, of course, be reformulated in terms of the critical equation. However, we postpone a discussion of the critical equation to a subsequent paper on d-dimensional Cartesian lattices in which only one coupling constant is extremely small. There we shall encounter a new aspect of critical equations, which is of direct importance for the asymptotic form of equation (1) for $t_2, t_3 \to 0$, so that a discussion of the critical equation of the s.c. lattice can be given a more firm basis there. In fact, we shall be led to a closed expression for the asymptotic form which is in good agreement with the numerical results obtained for b(a) in this paper.

REFERENCE

1) Citteur, C.A.W. and Kasteleyn, P.W., Physica 62 (1972) 17.

Note: The numbers b_{mn0} have also been calculated by a different method (to be presented in a subsequent chapter). The numerical values of the b_{mn0} given in table II are reproduced except that of $b_{32\ 0}$, which is found to be $10728\frac{8}{9}$. Since the second method is much less cumbersome than the one followed here, the latter value is accepted as the correct one.

CRITICAL BEHAVIOUR OF ISING SYSTEMS ON EXTREMELY ANISOTROPIC LATTICES

III. Generalization of Sykes' expression for the susceptibility and application to the ferromagnetic case

Synopsis

A new method for obtaining series expansions for the initial susceptibility of spin-$\frac{1}{2}$ Ising systems on extremely anisotropic lattices is presented. The method starts from a generalization to anisotropic lattices of an expression proposed by Sykes (J.math.Phys. 2 (1961) 52) for the susceptibility of isotropic lattices. As in the latter case, only closed graphs, i.e. graphs without points of valence 1, have to be counted; this means a substantial reduction in the labour involved in the derivation of series expansions in comparison with the method followed in two preceding papers in this series (I, II). By means of the new method the susceptibility series for the quadratic and the simple cubic lattice, obtained in these papers, are rederived; to the former series two more terms have been added.

§ 1. Introduction

In two preceding papers [1)2)], to be referred to as I and II henceforth, the critical behaviour of the initial susceptibility χ of ferromagnetic spin-$\frac{1}{2}$ Ising systems on extremely anisotropic lattices was studied. The present paper is devoted to a different (and more efficient) method for obtaining series expansions for χ in such cases. We set up a generalization to anisotropic lattices of an expression for χ, originally proposed by Sykes [3)4)], and later on proved by Nagle and Temperley [5)], for isotropic lattices. The advantage of Sykes' formula is that it gives χ in terms of closed graphs only (i.e. graphs without points of valence 1) with the additional property that there are two points of odd valence at most. These graphs are far less numerous than the magnetic graphs (i.e. graphs with exactly two points of odd valence), all of which have to be considered for the derivation of series expansions by means of the older method of Oguchi [6)]. Consequently, the possibility of making errors is much smaller in Sykes' method. A similar, and even more drastic, reduction in the process of counting graphs takes place in our case of extremely anisotropic lattices, if studied by means of the above-mentioned generalization. Furthermore, the generalized expression is a very convenient starting point for a discussion of antiferromagnetic Ising systems on extremely anisotropic lattices, which will be given in the next paper of this series.

The generalization of Sykes' expression for χ is derived in § 2. In § 3a we consider its asymptotic form in the case of an extremely anisotropic d-dimensional Cartesian lattice as introduced in I, § 1. The formalism of dividing 2-graphs into slices and pieces, which proved to be very useful for the calculations given in I and II, and which entered there in a natural way, can be used with only slight modifications. In § 3b we consider the extremely anisotropic quadratic lattice; owing to the reduction in the labour of counting graphs two extra terms can be added to the series obtained in I for the leading order term χ_0' in the reduced susceptibility $(kT/\mu^2)\chi \equiv \chi'$; as a result a somewhat better estimate of the radius of convergence of this series is found. In §3c, finally, the series obtained in II for the simple cubic (s.c.) lattice is rederived; for this lattice, too, the new method is much more efficient than the straightforward but cumbersome procedure followed in II. The paper ends with a short discussion on the advantages of the new method.

§ 2. Extension of Sykes' formula for the susceptibility to anisotropic lattices

In order to derive an expression, similar to that of Sykes [4], for the reduced initial susceptibility χ' of a spin-$\frac{1}{2}$ Ising system on an anisotropic lattice, we shall closely follow the method developed by Nagle and Temperley [5] for isotropic lattices. With a view to the fact that we shall apply the generalized formula to Cartesian lattices we shall restrict ourselves to lattices with the property that the set of lines can be split up into d classes in such a way that with each point of the lattice exactly two lines of each class are incident. To all lines within a given class the same coupling constant J_i is assigned, where J_i may vary with i. This partitioning may be carried through still further by a splitting of each class into two classes in such a way that with each point only one line of each class is incident; the values of the coupling constant for different classes may be different again. The formalism for this case, which can also be applied to e.g. an anisotropic honeycomb lattice, is similar to the procedure for the above-mentioned kind of anisotropy.

To any partial graph G of the lattice (which is supposed to contain N points) a vector $\vec{p}$ with 3^d components $p_{s_1 \ldots s_d}$ ($s_i = 0,1,2$ for $i=1,\ldots,d$) is assigned; $p_{s_1 \ldots s_d}$ is the number of vertices of the lattice with which exactly s_1 lines of the 1^{st} class, s_2 lines of the 2^{nd} class, etc. of G are incident. Denoting by $g_N(\vec{p})$ the number of partial graphs G of the lattice to which a given vector $\vec{p}$ is assigned, we can cast the partition function of the Ising system on this lattice in the presence of an external magnetic field H into the form

$$Z_N(\vec{z}) = \left[2 \cosh\beta\mu H \prod_{i=1}^{d} \cosh\beta J_i\right]^N \sum_{\vec{p}} g_N(\vec{p}) \prod_{s_1=0}^{2} \cdots \prod_{s_d=0}^{2} z_{s_1 \ldots s_d}^{p_{s_1 \ldots s_d}}, \tag{1}$$

where $\vec{z}$ is a vector with 3^d components, defined by

$$z_{s_1 \ldots s_d} = \begin{cases} \tau \prod_{i=1}^{d} t_i^{s_i/2} & \text{for } \Sigma_{i=1}^{d} s_i \text{ odd} \\[2ex] \prod_{i=1}^{d} t_i^{s_i/2} & \text{for } \Sigma_{i=1}^{d} s_i \text{ even} \end{cases} \tag{2}$$

with $\tau = \tanh\beta\mu H$, $t_i = \tanh\beta J_i$, $s_i = 0,1,2$.

A straightforward extension of the combinatorial proof by Nagle and Temperley in § 3 of ref. 5 yields the following functional equation for Z_N:

$$Z_N(\vec{z}) = \left[\prod_{i=1}^{d} (1+y_i^2)^{-N}\right] Z_N(\vec{v}), \tag{3}$$

where the y_i $(i=1,\ldots,d)$ are arbitrary numbers and $\vec{v}$ stands for a vector with 3^d components:

$$v_{s_1\ldots s_d} = \sum_{r_1=0}^{2} \cdots \sum_{r_d=0}^{2} z_{r_1\ldots r_d} \sum_{t_1=0}^{r_1} \cdots \sum_{t_d=0}^{r_d} \prod_{i=1}^{d} (-)^{t_i} \binom{2-s_i}{r_i-t_i}\binom{s_i}{t_i} y_i^{r_i+s_i-2t_i}. \tag{4}$$

As in ref. 4, the importance of this theorem is due to the fact that <u>for given $\vec{z}$</u> we can choose the numbers y_i such that only <u>closed</u> graphs give a non-zero contribution to $Z_N(\vec{v})$, in contrast to the situation for $Z_N(\vec{z})$, to which <u>all</u> partial graphs of the lattice contribute. It follows from the definition of Z_N that this happens if each of the d components $v_{10\ldots0}, v_{010\ldots0}, \ldots, v_{00\ldots01}$ is chosen to be zero; this choice implies a set of d equations from which the y_i have to be solved in terms of τ and $\vec{z}$. The Ising partition function can thus be expressed in terms of a (relatively small) subset of all partial graphs of the lattice. In order to employ this alternative expansion of the Ising partition function for actual calculations it is necessary to solve the equations for the y_i; for arbitrary values of the magnetic field H, however, this is not possible. Fortunately, as is shown below, the calculation of χ' requires only the solution in first order of H, which is readily found.

Rewriting the identity

$$\chi' = \lim_{N\to\infty} \frac{1}{N} \left[\frac{\partial^2}{\partial\tau^2} \log Z_N(\vec{z})\right]_{\tau=0}, \tag{5}$$

which can be derived immediately from the definition of χ', in terms of the y_i and $\vec{v}$ by means of eq. (3), one finds that these quantities need to be known up to second order in τ only. However, still further simplifications arise.

a) The y_i vanish linearly with τ:

$$y_i = -\frac{t_i^{\frac{1}{2}} D_i^{(d)}}{D^{(d)}}\,\tau + \mathcal{O}(\tau^2) \qquad (\tau\to0); \tag{6}$$

for the derivation and the notation see the appendix. Therefore, one may restrict oneself to those terms of the r.h.s. of eq. (4) that are of at most second order in the y_i.

b) In eq. (4) the term of zeroth order in the y_i is $(-)^{\Sigma_i s_i} z_{s_1 \ldots s_d}$, which for odd values of $\Sigma_i s_i$ vanishes linearly with τ (see eq. (2)). Consequently, if $\Sigma_i s_i$ is odd, $v_{s_1 \ldots s_d}$ itself vanishes linearly with τ. Since each vertex in a graph G yields a corresponding factor $v_{s_1 \ldots s_d}$ in the term of $Z_N(\vec{v})$ to which G contributes, the contribution from graphs with more than two vertices of odd valence is of more than second order in τ. Since for χ' only the contributions of second order in τ are relevant and since furthermore the number of vertices of odd valence in a graph is even, we conclude that only closed graphs with zero or two points of odd valence have to be taken into account for the calculation of χ'. It follows that for odd values of $\Sigma_i s_i$ the $v_{s_1 \ldots s_d}$ (and also the y_i) have to be known up to first order in τ only; substituting eq. (6) we find

$$v_{s_1 \ldots s_d} = \frac{\left(\prod_{i=1}^{d} t_i^{s_i/2}\right)\left(-D^{(d)} - \sum_{i=1}^{d} \left[s_i-(2-s_i)t_i \right] D_i^{(d)}\right)}{D^{(d)}} \, \tau + \mathcal{O}(\tau^2) \quad (\tau \to 0). \qquad \left(\sum_{i=1}^{d} s_i \text{ odd}\right) \tag{7}$$

c) For even values of $\Sigma_i s_i$, too, it is sufficient for the calculation of χ' to know the y_i only up to the first-order term in τ. Second-order terms would be relevant in those terms of the r.h.s. of eq. (4) which are linear in the y_i. However, all such terms contain a factor $z_{r_1 \ldots r_d}$ with $\Sigma_i r_i$ odd, because the number $\Sigma_i (r_i + s_i - 2t_i)$ is 1 for these terms and hence, if $\Sigma_i s_i$ is even, $\Sigma_i r_i$ is odd; as mentioned above such $z_{r_1 \ldots r_d}$ contain a factor τ, so that first-order terms in the y_i in eq. (4) actually yield terms of order τ^2. Substituting eq. (6) we find

$$\begin{aligned} v_{s_1 \ldots s_d} = \left(\prod_{i=1}^{d} t_i^{s_i/2}\right)\Bigg\{1 + \Bigg[\sum_{i=1}^{d} \frac{\left[s_i-(2-s_i)t_i\right] D_i^{(d)}}{D^{(d)}} \; + \\ + \sum_{i=1}^{d} \frac{\left[\binom{s_i}{2} - s_i(2-s_i)t_i + \binom{2-s_i}{2} t_i^2\right]\left(D_i^{(d)}\right)^2}{\left(D^{(d)}\right)^2} \; + \\ + \sum_{\substack{i,j=1 \\ i<j}}^{d} \frac{\left[s_i s_j - s_i(2-s_j)t_j - s_j(2-s_i)t_i + (2-s_i)(2-s_j)t_i t_j\right] D_i^{(d)} D_j^{(d)}}{\left(D^{(d)}\right)^2} \Bigg] \tau^2 + \mathcal{O}(\tau^3)\Bigg\} \\ \left(\sum_{i=1}^{d} s_i \text{ even}\right) \quad (\tau \to 0). \end{aligned} \tag{8}$$

d) Finally, according to eqs. (3) and (5) the factor $\Pi_{i=1}^{d}(1+y_i^2)^{-N}$ in eq. (3) has to be differentiated with respect to τ. Again, only the first-order term in τ of the y_i is relevant.

The further evaluation of χ' is carried out in the same way as in ref. (5). One finds

$$\chi' = A + B + C + D \tag{9}$$

where

$$A = \frac{(-)^d \prod_{i=1}^{d}(1+t_i)}{D^{(d)}}\ ; \tag{9a}$$

$$B = -2\sum_{i=1}^{d} \frac{(1-t_i^2)\left[D_i^{(d)}\right]^2 \sum_{\vec{p}}'\left[g(\vec{p})n_i \prod_{j=1}^{d} t_j^{n_j}\right]}{\left[D^{(d)}\right]^2} =$$

$$= -2\sum_{i=1}^{d} \frac{(1-t_i^2)\left[D_i^{(d)}\right]^2 t_i \frac{\partial}{\partial t_i} \sum_{\vec{p}}'\left[g(\vec{p}) \prod_{j=1}^{d} t_j^{n_j}\right]}{\left[D^{(d)}\right]^2}\ ; \tag{9b}$$

$$C = \frac{\left[\prod_{i=1}^{d}(1+t_i)^2\right] \sum_{\vec{p}}'\left\{g(\vec{p}) \sum_{s_1=0}^{2}\cdots\sum_{s_d=0}^{2}\left[p_{s_1\ldots s_d}\ s(s-2)\right] \prod_{j=1}^{d} t_j^{n_j}\right\}}{\left[D^{(d)}\right]^2}\ ; \tag{9c}$$

$$D = \frac{2\left[\prod_{i=1}^{d}(1+t_i)^2\right] \sum_{\vec{p}}''\left\{g(\vec{p})(s_P-1)(s_Q-1) \prod_{j=1}^{d} t_j^{n_j}\right\}}{\left[D^{(d)}\right]^2}. \tag{9d}$$

In these expressions $g(\vec{p})$ is the coefficient of N^1 in $g_N(\vec{p})$, s stands for $\Sigma_i s_i$, the valence of a point, and $n_j(=n_j(\vec{p})\,) = \frac{1}{2}\Sigma_{s_1}\ldots\Sigma_{s_d}\ s_j p_{s_1\ldots s_d}$ is the number of lines in a graph with a given $\vec{p}$ which belong to the j^{th} class. The summations in B and C are over all $\vec{p}$ assigned to graphs in which each point has even valence, the summation in D runs over all $\vec{p}$ assigned to graphs in which exactly two points, denoted by P and Q, have odd valence s_P and s_Q, respectively. By the presence in D of the factor $(s_P-1)(s_Q-1)$, only graphs in which both these valences differ from 1 give a non-zero contribution, as it should be, according to our construction of the y_i. Analogously, the graphs contributing to C should have at least one point with even valence larger than 2.

From the remarks made above it follows that the graphs considered in B, C and D have the property in common that they contain one or more polygons as a subgraph. If one would leave out these graphs from the evaluation of χ', an expression would result which is exact for a lattice with the same coordination number as the original lattice, but without polygons as subgraphs, a so-called Bethe lattice.[7] The Bethe-Peierls approximation for the original lattice is known to be an exact theory for this Bethe lattice; we therefore conclude that A is the Bethe-Peierls approximation for χ', which indeed can be verified by using a formalism developed by Fisher.[8] The term B is closely related to the nearest-neighbour spin-spin correlation functions (and hence to the internal energy), viz. in the following way. From eqs. (1) and (2) we deduce

$$\sum_{\vec{p}}{}' \left[g(\vec{p}) \prod_{j=1}^{d} t_j^{n_j} \right] = \log Z - \log\left[2 \prod_{j=1}^{d} \cosh\beta J_j \right] , \tag{10}$$

in which Z is the thermodynamic limit of the zero-field partition function per point of the lattice. Differentiating this identity with respect to t_i we find

$$\frac{\partial}{\partial t_i} \sum_{\vec{p}}{}' \left[g(\vec{p}) \prod_{j=1}^{d} t_j^{n_j} \right] = \frac{1}{1-t_i^2} \left[\frac{\partial \log Z}{\partial(\beta J_i)} - t_i \right] = \frac{1}{1-t_i^2} \left[\langle \sigma_{\vec{a}} \sigma_{\vec{a}+\vec{u}_i} \rangle - t_i \right], \tag{11}$$

in which $\langle \sigma_{\vec{a}} \sigma_{\vec{a}+\vec{u}_i} \rangle$ is the correlation function for two spins which are located on neighbouring points $\vec{a}$ and $\vec{a}+\vec{u}_i$ connected by a line with coupling constant J_i. Especially the last member of eq. (11) will turn out to play a central role in the susceptibility of an antiferromagnetic Ising system on an extremely anisotropic Cartesian lattice, which will be discussed in the following paper of this series.

§ 3. Extremely anisotropic lattices (ferromagnetic case).

a) Cartesian lattices.

We now consider an Ising system on a d-dimensional Cartesian lattice as introduced in I, § 1, for which $0<J_i<<J_1$ $(i=2,\dots,d)$. The classes into

which the set of lines is split up correspond to the sets of lines parallel to the respective lattice axes; hence, d is the dimensionality of the lattice. For the evaluation of the terms B, C and D it is convenient to change the summations over $\vec{p}$ into summations over partial graphs of the lattice. The requirement that one has to count graphs <u>per lattice point</u> (which is implied by the definition of $g(\vec{p})$) is then reflected in the fact that the summations run over <u>overlappings</u> formed by one or more line components of type 0 in B and C, and by one line component of type 2 and zero, one, two, ... line components of type 0 in D (called 2-graphs in I). Since in the above-mentioned case of extreme anisotropy of the lattice the critical behaviour takes place for $(t_1,t_2,\ldots,t_d) \approx (1,0,\ldots,0)$, we rewrite each one of the quantities A, B, C and D as a multiple power series in $t_2,\ldots,t_d$ with coefficients depending on t_1. Generalizing the division of 2-graphs into pieces, as described in I and II, to the overlappings considered here we find again that these coefficients are sums of terms which are infinite for $t_1=1$; as before, we assume that for the investigation of the asymptotic form of the critical equation one may restrict oneself to those contributions to the coefficients of the resulting series $\Sigma_{n_2=0}^{\infty}\cdots\Sigma_{n_d=0}^{\infty}\, a_{n_2\ldots n_d}(t_1)\, t_2^{n_2}\cdots t_d^{n_d}$ for χ' that have the strongest divergence for $t_1\to 1$. On account of the results for the quadratic and the simple cubic lattice we expect that for $a_{n_2\ldots n_d}(t_1)$ these contributions are proportional to $(1-t_1)^{-(n_2+\ldots+n_d+1)}$ in this limit. This is true indeed for A, the Bethe-Peierls approximation for χ'. Using eq. (A6) of the appendix we rewrite A as

$$A = \frac{1}{1-2\sum_{i=1}^{d} t_i/(1+t_i)}\ . \tag{12}$$

Treating this expression in the same way as the r.h.s. of eq. (A6) of the appendix we find that the most divergent contributions to the coefficients of the corresponding multiple power series in $t_2, \ldots, t_d$ come from the terms found by expanding the second factor of the quantity

$$A_0 \equiv \frac{1}{1-t_1}\cdot\frac{2}{1-4t_2/(1-t_1)-\ldots-4t_d/(1-t_1)}\ , \tag{13}$$

which shows that these contributions have the required property.

For the investigation of B we rewrite it as

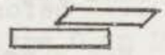
$$B = -2 \sum_{i=1}^{d} \frac{(1-t_i^2)\left[D_i^{(d)}\right]^2 t_i \frac{\partial}{\partial t_i} \sum_0{}' \left[c(0) \prod_{j=1}^{d} t_j^{n_j}\right]}{\left[D^{(d)}\right]^2} ; \tag{14}$$

the inner summation is over all overlappings 0 consisting of line components of type 0, $n_j (=n_j(0))$ is the number of lines of 0 parallel to the j^{th} lattice axis; the factors $c(0)$ are of a similar nature and can be found in the same way as the factors +1, -1, 2, ... before the respective bracket expressions in the r.h.s. of I, eq. (30). For given values of $n_2, \ldots, n_d$ there are at most $(\Sigma_{j=2}^{d} n_j)-1$ pieces in the overlappings contributing to eq. (14). Differentiation with respect to t_1 of a contribution to the inner sum from an overlapping with this maximal number of pieces yields, among others, a term proportional to $(1-t_1)^{-(n_2+\ldots+n_d)}$; using eq. (A9) for $D^{(d)}$ and taking into account the factor $(1-t_1^2)$ in the numerator of the term corresponding to i=1 in eq. (14) we find that this term also yields a contribution to χ' with the required number $(\Sigma_{j=2}^{d} n_j)+1$ of factors $(1-t_1)^{-1}$. Differentiation with respect to the other t_i does, of course, not increase the power of $(1-t_1)^{-1}$ in a contribution to the inner sum in eq. (14), as was the case for i=1. On the other hand, the factor $(1-t_i^2)$ remains finite for $(t_1, t_2, \ldots, t_d) \to (1,0,\ldots,0)$, $i \neq 1$, so that there is again a contribution to χ' with the required number of factors $(1-t_1)^{-1}$. In a similar way it follows that the contributions to χ' from overlappings which contain less than the maximal possible number of pieces do not have enough factors $(1-t_1)^{-1}$ to make them of the type considered here; therefore, we will not take them into account.

The quantity C may be left out of consideration completely, since it yields contributions in which the power of $(1-t_1)^{-1}$ does not exceed the number $\Sigma_{j=2}^{d} n_j$ for given values of n_j. This is due to the fact that the overlappings which give a non-zero contribution to C are such that one or more of their constituting line components have a point of (even) valence larger than 2, so that there are $(\Sigma_{j=2}^{d} n_j)-2$ pieces at most. Consequently, in the contributions to χ' there is at least one factor $(1-t_1)^{-1}$ too few for making these contributions of the required form.

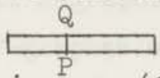
For D the graphs yielding enough factors $(1-t_1)^{-1}$ in the contributions to χ' are such that the valence of both points P and Q is 3 and the valence of all other points 2, which implies that the line components containing P and Q are dumbbells or star figure-eights in the terminology of Sykes.[4)] In this case the number of pieces in the overlappings may take the value

$(\Sigma_{j=2}^{d} n_j)-1$, which in combination with the denominator $\left[D^{(d)}\right]^2$ in eq. (9d) gives the required number of factors $(1-t_1)^{-1}$. For all other graphs, e.g. where s_P or s_Q is 5,7,..., there are less factors $(1-t_1)^{-1}$ present in the contribution to χ', so that again they need not be considered here.

Finally, we present the procedure for the actual calculation of B and D, introducing it by two examples. For B we consider the class of overlappings formed by two rectangles on the s.c. lattice, one of which contains two y-lines, the other one two z-lines, which can, by the rules given in II, table I, be represented by the diagram  . For this class, the number c(0) is -1 and the number of pieces is 3. This class contributes to the inner sum in the r.h.s. of eq. (14) a term $-t_2^2t_3^2[t_1^2/(1-t_1^2)]^2[t_1^4/(1-t_1^4)] = -t_2^2t_3^2t_1^8/(1-t_1^2)^2(1-t_1^4)$. The operator $t_1\partial/\partial t_1$ acting on this term may be replaced by a differentiation of the denominator alone, because correction terms to this contain fewer factors $(1-t_1)^{-1}$; replacing in the resulting quantity factors like t_1^p by 1 and $1-t_1^p$ by $p(1-t_1)$ (see I, § 4) we find in leading order in $(1-t_1)^{-1}$ a term $-3t_2^2t_3^2/16(1-t_1)^4$. Furthermore, we substitute for $D^{(3)}$ the expression given in eq. (A9) of the appendix and for $D_1^{(3)}$ the value assumed by the r.h.s. of eq. (A5) for $t_2=t_3=0$, i.e. 1; the contribution to χ' is then in leading order found to be $[3t_2^2t_3^2/4(1-t_1)^5][1+8t_2/(1-t_1) + 8t_3/(1-t_1) + \ldots]$. Similarly, the terms resulting from the effect of the operators $t_2\partial/\partial t_2$ and $t_3\partial/\partial t_3$ on $-t_2^2t_3^2t_1^8/(1-t_1^2)^2(1-t_1^4)$ are in leading order both found to be $-t_2^2t_3^2/8(1-t_1)^3$. Putting the factors $1-t_i^2$ and $D_i^{(3)}$ (i=2,3) in eq. (14) equal to 1 and 2, their respective values for $(t_1,t_2,t_3) = (1,0,0)$, we find for the resulting contribution to χ' both for i=2 and i=3 the quantity $[t_2^2t_3^2/(1-t_1)^5][1+8t_2/(1-t_1)+8t_3/(1-t_1)+\ldots]$.

From this example the procedure to be followed for an arbitrary class of overlappings in B for d=2,3,... becomes clear: in its contribution to the inner sum in eq. (14) the differentiation with respect to t_1 needs to be carried out only for the denominator; in the resulting quantity (<u>not</u> in the contribution itself) factors like $1-t_1^p$ have to be replaced by $p(1-t_1)$; for the other t_i the order of the differentiation and this replacement is irrelevant. The substitution for quantities in the remaining part of eq. (14) is as follows:

$$\left.\begin{array}{l} 1-t_1^2 \rightarrow 2(1-t_1) \\ 1-t_i^2 \rightarrow 1, \quad i=2,\ldots,d \\ D_1^{(d)}(t_1,t_2,\ldots,t_d) \rightarrow D_1^{(d)}(1,0,\ldots,0) = (-)^{d-1} \\ D_i^{(d)}(t_1,t_2,\ldots,t_d) \rightarrow D_i^{(d)}(1,0,\ldots,0) = (-)^{d-1}\,2, \quad i=2,\ldots,d \\ D^{(d)}(t_1,t_2,\ldots,t_d) \rightarrow (-)^d(1-t_1)\left[1-4\sum_{i=2}^{d} t_i/(1-t_1)\right] \end{array}\right\} \quad (15)$$

For D we take as an example the simplest class of 2-graphs possible, viz. the class  on the quadratic lattice. Since $s_P=s_Q=3$, its contribution to the sum in eq. (9d) is $4t_2^3t_1^4/(1-t_1^2)^2$, of which the leading order term for $t_1\to 1$ is $t_2^3/(1-t_1)^2$. Taking for the factor $\Pi_{i=1}^{2}(1+t_i)^2$ its value assumed for $(t_1,t_2)=(1,0)$, i.e. 4, and using eq. (15) for $D^{(2)}$ we find in leading order the contribution to χ': $[8t_2^3/(1-t_1)^4][1+8t_2/(1-t_1)+\dots]$.

Generally, since for all graphs under consideration s_P and s_Q have to be 3, we may take the factor $(s_P-1)(s_Q-1)$ outside the summation, multiply it by the value of $2\Pi_{i=1}^{d}(1+t_i)^2$ for $(t_1,t_2,\dots,t_d)=(1,0,\dots,0)$, i.e. 8, substitute eq. (A9), and thus replace D by the expression D_0 defined by

$$D_0 \equiv \frac{32\sum\limits_{n_2=0}^{\infty}\cdots\sum\limits_{n_d=0}^{\infty} d_{n_2\dots n_d}\,[t_2/(1-t_1)]^{n_2}\dots[t_d/(1-t_1)]^{n_d}}{(1-t_1)[1-4t_2/(1-t_1)-\dots-4t_d/(1-t_1)]^2}. \tag{16}$$

The numbers $d_{n_2\dots n_d}$ consist of contributions from all classes of 2-graphs with n_i lines with coupling constant J_i, $i=2,\dots,d$, for which $s_P=s_Q=3$ and for which the number of pieces is $(\Sigma_i n_i)-1$. The example treated here yields a contribution to d_3, viz. $+1\cdot\frac{1}{4}$, the factor +1 arising from the fact that we have a 2-graph consisting of a single line component, the factor $\frac{1}{4}$ being due to the presence of exactly two pieces, each with multiplicity $\frac{1}{2}$. Generally, the formalism introduced in I, especially the concepts of piece, slice and weight, may here be used to their full extent.

b) <u>The quadratic lattice</u>

The evaluation of B for this lattice does not require an expansion in terms of graphs, because the partition function Z is known in closed form [9]:

$$\log Z = \log 2 + \frac{1}{8\pi^2}\int_{-\pi}^{+\pi} d\omega_1 \int_{-\pi}^{+\pi} d\omega_2\, \log(\cosh 2\beta J_1\cosh 2\beta J_2 - \sinh 2\beta J_1\cos\omega_1 - \sinh 2\beta J_2\cos\omega_2). \tag{17}$$

Combining this equation with eqs. (9b) and (11) we find

$$B = -2\sum_{i=1}^{2}\frac{\left(D_i^{(2)}\right)^2 t_i\left\{\frac{1}{4\pi^2}\int_{-\pi}^{+\pi} d\omega_1\int_{-\pi}^{+\pi} d\omega_2\,\frac{2t_i(1+t_{i'}^2)-(1+t_i^2)(1-t_{i'}^2)\cos\omega_i}{(1+t_1^2)(1+t_2^2)-2t_1(1-t_2^2)\cos\omega_1-2t_2(1-t_1^2)\cos\omega_2}-t_i\right\}}{\left(D^{(2)}\right)^2}, \tag{18}$$

where $i'=2$ for $i=1$ and vice versa.

The term in eq. (18) which corresponds to $i=1$ can be written as

$$\frac{-2\left[D_1^{(2)}\right]^2 t_1(1-t_1^2)}{\left[D^{(2)}\right]^2}\frac{1}{4\pi^2}\int_{-\pi}^{+\pi}d\omega_1\int_{-\pi}^{+\pi}d\omega_2\,\frac{t_1(1+t_2^2)-(1-t_2^2)\cos\omega_1+2t_1t_2\cos\omega_2}{(1+t_1^2)(1+t_2^2)-2t_1(1-t_2^2)\cos\omega_1-2t_2(1-t_1^2)\cos\omega_2}\,.$$

The function multiplying the integral becomes proportional to $(1-t_1)^{-1}\,[1-4t_2/(1-t_1)]^{-2}$ for $(t_1,t_2)\to(1,0)$; noting that this expression contains a factor $(1-t_1)^{-1}$ and combining this fact with the remarks about B made in the previous section, we expect that for $(t_1,t_2)\to(1,0)$ the integral itself approaches a finite non-zero value, which may depend on $t_2/(1-t_1)$. In order to find this function we change from the variables t_1 and t_2 to the variables $1-t_1$ and $t_2/(1-t_1)$ in the integrand; next we rewrite the numerator and the denominator as polynomials in $1-t_1$ with coefficients depending on $t_2/(1-t_1)$, ω_1 and ω_2. The term of zeroth order in $(1-t_1)$ in these polynomials is $1-\cos\omega_1$ and $2(1-\cos\omega_1)$ respectively, so that special care has to be taken for the case $\omega_1=0$. The remaining procedure is standard: one puts the variable $1-t_1$ equal to zero except for some small interval around $\omega_1=0$, where $\cos\omega_1$ is replaced by $1-(\omega_1^2/2)$. In an analogous way the leading order term for $i=2$ can be found. After performing the integration over ω_1 and writing ω for ω_2 one finds the resulting expression for the leading order term of B itself, to be denoted by B_0:

$$B_0 = \frac{2}{(1-t_1)(1-4x)^2}\left[\frac{1}{2\pi}\int_{-\pi}^{+\pi}d\omega\,\frac{1-2x\cos\omega}{(1-4x\cos\omega+4x^2)^{\frac{1}{2}}}-1\right]+$$

$$+\frac{8x}{(1-t_1)(1-4x)^2}\,\frac{1}{2\pi}\int_{-\pi}^{+\pi}d\omega\,\frac{\cos\omega-2x}{(1-4x\cos\omega+4x^2)^{\frac{1}{2}}}\,;\qquad x\equiv\frac{t_2}{1-t_1}\,; \qquad (19)$$

here the first and second term are the leading order terms corresponding to $i=1$ and 2, respectively, in eq. (18). The first few coefficients of the resulting power series in x are listed in table II.

The quantity D_0 cannot be calculated explicitly, we can only find the first few numbers d_n (we write n instead of n_2) occurring in eq. (16) by counting all 2-graphs which contribute to these numbers. The book-keeping for these 2-graphs is similar to the system used in I; it was found convenient to order the dumbbells  according to the pair of polygons which result after the "bridge" s has been erased, and to order the star figure-eights  according to the "external" polygon which is left after the "internal" bridge s has been erased. The book-keeping for the bridges s is analogous to that used for self-avoiding walks in I. The results for d_n, n=0,1,...,7,are listed in table I.

TABLE I

The coefficients d_n, n=0,1,...,7,together with the contributions from the various types of 2-graphs to these coefficients. The meaning of the symbols 2,2-0, etc. is the same as in I. As mentioned in §3a the line components of type 2 are dumbbells or star figure-eights.

Type of 2-graphs	d_0	d_1	d_2	d_3	d_4	d_5	d_6	d_7
2	-	-	-	$\frac{1}{4}$	$1\frac{1}{3}$	$4\frac{43}{48}$	$16\frac{5}{108}$	$49\frac{5069}{5184}$
2-0	-	-	-	-	-	$-\frac{9}{16}$	$-4\frac{37}{72}$	$-22\frac{19}{24}$
0-2-0	-	-	-	-	-	-	-	$\frac{1}{2}$
$\overset{2}{\text{0-0}}$	-	-	-	-	-	-	-	$\frac{35}{64}$
2-0-0	-	-	-	-	-	-	-	$\frac{7}{16}$
	0	0	0	$\frac{1}{4}$	$1\frac{1}{3}$	$4\frac{1}{3}$	$11\frac{115}{216}$	$28\frac{869}{1296}$

The coefficients up to 7^{th} order of the series expansion for $(1-t_1)D_0$, which follow from these numbers,are listed in table II, together with the coefficients for $(1-t_1)A_0$ (see eq. (13)) and $(1-t_1)B_0$ and the resulting coefficients for $(1-t_1)X_0'$, where X_0' is written as in I as

$$X_0' = \frac{1}{1-t_1} \sum_{n=0}^{\infty} b_{n0} \left(\frac{t_2}{1-t_1}\right)^n .$$

TABLE II

The coefficients b_{n0}, $0 \leq n \leq 7$, together with the contributions from the quantities A_0, B_0 and D_0 to these coefficients.

	b_{00}	b_{10}	b_{20}	b_{30}	b_{40}	b_{50}	b_{60}	b_{70}
A_0	2	8	32	128	512	2048	8192	32768
B_0	-	-	-10	- 80	$-485\frac{1}{2}$	-2604	$-13072\frac{1}{2}$	-62916
D_0	-	-	-	8	$106\frac{2}{3}$	864	$5574\frac{10}{27}$	$31688\frac{34}{81}$
X_0'	2	8	22	56	$133\frac{1}{6}$	308	$693\frac{47}{54}$	$1540\frac{34}{81}$

The newly obtained coefficients b_{60} and b_{70} of the series $\Sigma_{n=0}^{\infty} b_{n0} x^n$ should provide a more reliable estimate of the quantities b and p describing the behaviour of the series near its radius of convergence b^{-1}:

$$\sum_{n=0}^{\infty} b_{n0} x^n \sim (1-bx)^{-p} \qquad (x \uparrow b^{-1}). \tag{20}$$

Indeed, carrying out an analysis of the series along the same lines as in I, § 6 one finds that the new intermediate results for b and, to a less extent, for p lie closer to and seem to converge better to the conjectured respective values 2 and 7/4 than the old ones. This fact adds to the confidence one may have in the procedure for the investigation of the asymptotic form of the critical equation on which I and II are founded. Our new estimates are

$$\begin{aligned} b &= 1.99 \pm 0.02 \\ p &= 1.77 \pm 0.03 . \end{aligned} \tag{21}$$

Putting p equal to the conjectured value 7/4 we find a somewhat higher estimate for b than in eq. (21):

$$b = 2.01 \pm 0.01. \tag{22}$$

On the other hand, putting b equal to 2 we find for p:

$$p = 1.77 \pm 0.02. \tag{23}$$

c) The simple cubic lattice.

For the s.c. lattice both quantities B_0 and D_0 have to be evaluated by means of a graphical expansion.

i) Taking into account only the overlappings with the required number of pieces the leading order term of the quantity $\Sigma_0'\left(c(0) \prod_{j=1}^{3} t_j^{n_j}\right)$ in eq. (14) is found to be:

$$\frac{t_1^2}{1-t_1^2}\left(t_2^2+t_3^2\right) + \frac{4t_1^6}{(1-t_1^2)^3}\left(t_2^4+t_3^4\right) + \left[\frac{32t_1^6}{(1-t_1^2)^3} + \frac{8t_1^8}{(1-t_1^2)^2(1-t_1^4)}\right]t_2^2t_3^2 + \ldots -$$

$$- \left\{\frac{t_1^8}{(1-t_1^2)^2(1-t_1^4)}\left(6t_2^4+6t_3^4+16t_2^2t_3^2\right) + \ldots\right\} + \ldots ;$$

the terms occurring with a + sign arise from overlappings that consist of a single polygon, those with a - sign from overlappings formed by two polygons. Substituting this expansion into eq. (14) and using also eq. (15) one finds the numbers listed for B_0 in table IV at the end of this section.

ii) The first few numbers d_{mn} occurring in the expression for D_0 (eq. (16)) are listed in table III.

TABLE III

The coefficients d_{mn}, $0 \leq m,n \leq 5$, $0 \leq m+n \leq 5$ together with the contributions from the various types of 2-graphs to these coefficients. For symmetry reasons only the d_{mn} with $m \geq n$ are given.

Type of 2-graphs	d_{00}	d_{10}	d_{20}	d_{11}	d_{30}	d_{21}	d_{40}	d_{31}	d_{22}	d_{50}	d_{41}	d_{32}
2	-	-	-	-	$\frac{1}{4}$	-	$1\frac{1}{3}$	-	$3\frac{1}{3}$	$4\frac{43}{48}$	$6\frac{1}{4}$	$15\frac{31}{36}$
2-0	-	-	-	-	-	-	-	-	-	$-\frac{9}{16}$	-	$-\frac{3}{4}$
	0	0	0	0	$\frac{1}{4}$	0	$1\frac{1}{3}$	0	$3\frac{1}{3}$	$4\frac{1}{3}$	$6\frac{1}{4}$	$15\frac{1}{9}$

The results for $(1-t_1)D_0$ following from these numbers d_{mn} are listed in table IV, together with the coefficients for $(1-t_1)A_0$ (see eq. (13)) and $(1-t_1)B_0$ and the resulting coefficients for $(1-t_1)\chi_0'$, where χ_0' is written as in II as

$$\chi_0' = \frac{1}{1-t_1} \sum_{m,n=0}^{\infty} b_{mn0} \left(\frac{t_2}{1-t_1}\right)^m \left(\frac{t_3}{1-t_1}\right)^n .$$

TABLE IV

The coefficients b_{mn0}, $0 \leq m,n \leq 5$, $0 \leq m+n \leq 5$, together with the contributions from the quantities A_0, B_0 and D_0. For symmetry reasons only the b_{mn0} with $m \geq n$ are given.

	b_{000}	b_{100}	b_{200}	b_{110}	b_{300}	b_{210}	b_{400}	b_{310}	b_{220}	b_{500}	b_{410}	b_{320}
A_0	2	8	32	64	128	384	512	2048	3072	2048	10240	20480
B_0	-	-	-10	-	-80	-80	$-485\frac{1}{2}$	-960	-1114	-2604	-7724	-11472
D_0	-	-	-	-	8	-	$106\frac{2}{3}$	64	$106\frac{2}{3}$	864	$1309\frac{1}{3}$	$1720\frac{8}{9}$
χ_0'	2	8	22	64	56	304	$133\frac{1}{6}$	1152	$2064\frac{2}{3}$	308	$3825\frac{1}{3}$	$10728\frac{8}{9}$

§ 4. Discussion

The generalized expression for the susceptibility of a spin-$\frac{1}{2}$ Ising system on a d-dimensional Cartesian lattice as derived in the present paper has proved to be very useful for our case of extreme anisotropy. This fact is illustrated by the following points.

a) The labour involved in counting graphs was substantially reduced, according to a rough estimate even by a factor eight. This reduction is due to the fact that the graphs in question are both simpler in structure and less numerous than the graphs which had to be taken into account in I and II. For the quadratic lattice an additional advantage is present, viz. the fact that the term B (and hence B_0) is known in closed form, so that a detailed expansion in terms of graphs can be avoided. By these two facts two more terms for the susceptibility of the quadratic lattice could be obtained. As to the s.c. lattice, it is gratifying that, apart from b_{320}, all coefficients as derived in II were reproduced. Since errors had much less chance of creeping into the calculations than in II, we have not reconsidered all calculations performed there for b_{320}, but instead we take the value which has been found in the present paper, as the correct one.

b) The case of extreme anisotropy turned out to yield an additional simplification, which did not occur in the isotropic case, viz. the quantity C is zero in leading order and hence does not play any role.

c) The generalized formula yields automatically a closed expression for the susceptibility in the Bethe-Peierls approximation of an anisotropic lattice with d lattice axes.

APPENDIX

In this appendix we discuss the solution of the y_i which makes the components $v_{10\cdots0}$, $v_{010\cdots0}$, ..., $v_{00\cdots01}$ of the vector $\vec{v}$ equal to zero. Expanding eq. (4) for these components up to first order in the y_i and putting them equal to zero we obtain

$$0 = v_{10\cdots0} = -z_{10\cdots0} - \{(z_{20\cdots0} - z_{00\cdots0})y_1 + 2z_{110\cdots0}\,y_2 + 2z_{1010\cdots0}\,y_3 + \ldots + 2z_{10\cdots01}\,y_d\} + \ldots , \tag{A1}$$

and analogous expressions for $v_{010\cdots0}$ etc. After substitution of eq. (2) we find the following equations for the y_i (again up to first order):

$$\left.\begin{array}{l}(t_1-1)y_1 + 2t_1^{\frac12}t_2^{\frac12}y_2 + 2t_1^{\frac12}t_3^{\frac12}y_3 + \ldots + 2t_1^{\frac12}t_d^{\frac12}y_d = -\tau t_1^{\frac12} ,\\ 2t_2^{\frac12}t_1^{\frac12}y_1 + (t_2-1)y_2 + 2t_2^{\frac12}t_3^{\frac12}y_3 + \ldots + 2t_2^{\frac12}t_d^{\frac12}y_d = -\tau t_2^{\frac12} ,\\ \quad\vdots \\ 2t_d^{\frac12}t_1^{\frac12}y_1 + 2t_d^{\frac12}t_2^{\frac12}y_2 + 2t_d^{\frac12}t_3^{\frac12}y_3 + \ldots + (t_d-1)y_d = -\tau t_d^{\frac12} .\end{array}\right\} \tag{A2}$$

By the application of Cramer's rule the solution of this set of linear equations is found to be

$$y_i = -\frac{t_i^{\frac{1}{2}} D_i^{(d)}}{D^{(d)}}\,\tau, \qquad (i = 1,\ldots,d) \tag{A3}$$

in which $D^{(d)}$ and $D_i^{(d)}$ are defined as follows:

$$D^{(d)} \equiv \begin{vmatrix} (t_1-1) & 2t_1^{\frac{1}{2}}t_2^{\frac{1}{2}} & \cdot & \cdot & \cdot & 2t_1^{\frac{1}{2}}t_d^{\frac{1}{2}} \\ 2t_2^{\frac{1}{2}}t_1^{\frac{1}{2}} & (t_2-1) & \cdot & \cdot & \cdot & 2t_2^{\frac{1}{2}}t_d^{\frac{1}{2}} \\ \cdot & \cdot & & & & \\ \cdot & \cdot & & & & \\ \cdot & \cdot & & & & \\ 2t_d^{\frac{1}{2}}t_1^{\frac{1}{2}} & 2t_d^{\frac{1}{2}}t_2^{\frac{1}{2}} & \cdot & \cdot & \cdot & (t_d-1) \end{vmatrix}, \tag{A4a}$$

$$D_i^{(d)} \equiv \begin{vmatrix} & & t_1^{\frac{1}{2}} & & \\ & & t_2^{\frac{1}{2}} & & \\ & & \cdot & & \\ & & \cdot & & \\ & & \cdot & & \\ & & t_{i-1}^{\frac{1}{2}} & & \\ 2t_1^{\frac{1}{2}} & 2t_2^{\frac{1}{2}}\ldots 2t_{i-1}^{\frac{1}{2}} & 1 & 2t_{i+1}^{\frac{1}{2}}\ldots 2t_d^{\frac{1}{2}} & \\ & & t_{i+1}^{\frac{1}{2}} & & \\ & & \cdot & & \\ & & \cdot & & \\ & & \cdot & & \\ & & t_d^{\frac{1}{2}} & & \end{vmatrix}; \tag{A4b}$$

the determinant for $D_i^{(d)}$ is understood to be the same as the one for $D^{(d)}$ except for the i^{th} row and the i^{th} column which are therefore specified.

Subtracting, for each $j \neq i$, the i^{th} row of the determinant $D_i^{(d)}$, multiplied by $t_j^{\frac{1}{2}}$, from the j^{th} row we obtain a determinant in which the only non-zero elements are found in the (unchanged) i^{th} row and on the main diagonal, the j^{th}

diagonal element being equal to $-1-t_j$, $j\neq i$. It follows that

$$D_i^{(d)} = (-)^{d-1} \prod_{\substack{j=1\\ j\neq i}}^{d} (1+t_j). \tag{A5}$$

For the evaluation of $D^{(d)}$ we write its diagonal elements as $2t_i^{\frac{1}{2}}t_i^{\frac{1}{2}}-(1+t_i)$, $i=1,\ldots,d$. Next we expand $D^{(d)}$ into a sum of terms to which d, d-1, d-2, ... diagonal elements have contributed a factor $(1+t_i)$, in the same way as in the eigenvalue problem for an arbitrary matrix A one expands the determinant $|A-\lambda I|$, where I is the unit matrix, into a polynomial in λ. This expansion is convenient because only the products formed by d and d-1 factors $(1+t_i)$ are multiplied by a non-vanishing factor. Products which consist of d-k factors $(1+t_i)$ are multiplied by principal $k\times k$ minors of the matrix obtained from $D^{(d)}$ by leaving out the terms $-(1+t_i)$ from its diagonal elements; for $2\leq k\leq d$ these minors all vanish because of the proportionality of their rows (and columns). In this way we find

$$D^{(d)} = (-)^d \prod_{j=1}^{d} (1+t_j) + \sum_{i=1}^{d} \left[(-)^{d-1}\, 2t_i \prod_{\substack{j=1\\ j\neq i}}^{d} (1+t_j)\right] =$$

$$(-)^d \left[\prod_{j=1}^{d} (1+t_j)\right]\left[1 - \sum_{i=1}^{d} \frac{2t_i}{1+t_i}\right] ; \tag{A6}$$

this implies in combination with eq. (A5) another identity

$$D^{(d)} = (-)^d \prod_{j=1}^{d} (1+t_j) + \sum_{j=1}^{d} 2t_j D_j^{(d)} \quad , \tag{A7}$$

which is very useful in the derivation of the expressions for A through D in eqs. (9a-d).

For the case of extreme anisotropy we write eq. (A6) as

$$D^{(d)} = (-)^d \left[\prod_{j=1}^{d} (1+t_j)\right]\left[\frac{1-t_1}{1+t_1} - \frac{2t_2}{1+t_2} - \ldots - \frac{2t_d}{1+t_d}\right] =$$

$$(-)^d \left[\prod_{j=1}^{d} (1+t_j)\right] \frac{1-t_1}{1+t_1} \left[1 - \frac{2t_2(1+t_1)}{(1-t_1)(1+t_2)} - \ldots - \frac{2t_d(1+t_1)}{(1-t_1)(1+t_d)}\right], \tag{A8}$$

which implies that in leading order

$$D^{(d)} \simeq (-)^d(1-t_1)\left(1-\frac{4t_2}{1-t_1}-\ldots-\frac{4t_d}{1-t_1}\right). \qquad \text{(A9)}$$

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CRITICAL BEHAVIOUR OF ISING SYSTEMS ON EXTREMELY ANISOTROPIC LATTICES

IV. Antiferromagnetic cases

Synopsis

The critical behaviour of the reduced magnetic susceptibility $\chi' \equiv (kT/\mu^2)\chi$ of spin-$\frac{1}{2}$ Ising systems on d-dimensional Cartesian lattices in which the coupling constants along d-1 lattice axes, $J_2,\ldots,J_d$, are much smaller than the coupling constant along the remaining lattice axis, J_1, and in which some of the J_i are negative, is investigated on the basis of the series expansion for χ' in the variables $t_i = \tanh\beta J_i$, $i=1,\ldots,d$. For $1+t_1 \ll 1$, $|t_2|,\ldots,|t_d| \ll 1$, χ' is found to behave as

$$\chi'(t_1,\ldots,t_d) \approxeq \chi_0'(t_1,\ldots,t_d) \equiv (1+t_1) \sum_{n_2=0}^{\infty} \cdots \sum_{n_d=0}^{\infty} c_{n_2\ldots n_d 0} \prod_{i=2}^{d} \left(\frac{t_i}{1+t_1}\right)^{n_i},$$

where the $c_{n_2\ldots n_d 0}$ are constants. In the limit $(t_1,\ldots,t_d) \to (-1,0,\ldots,0)$, taken in such a way that the power series in this expression remains convergent, χ_0' behaves through the factor $1+t_1$ as the susceptibility of the antiferromagnetic linear Ising chain. The d-dimensional nature of the system is contained in the power series.

It is rigorously shown that

$$\chi_0' = \tfrac{1}{2}(1 + \langle\sigma_1\sigma_2\rangle)_0 ,$$

where $\langle\sigma_1\sigma_2\rangle$ is the spin-spin correlation function for nearest neighbours along the first lattice axis, and $(\quad)_0$ is the leading-order term for $(t_1,t_2,\ldots,t_d) \to (-1,0,\ldots,0)$ of the quantity within the brackets. Using this relation it is shown that for the quadratic lattice with $(t_1,t_2) \approxeq (-1,0)$ χ_0' is given by

$$\chi_0' = (1+t_1)\, \frac{1}{\pi} \int_0^{\pi/2} \left[1-4\left(\frac{t_2}{1+t_1}\right)^2 \sin^2\psi\right]^{\frac{1}{2}} d\psi .$$

The fact that the integral, and hence χ_0', displays the same critical behaviour

as the internal energy is discussed in connection with the critical behaviour of χ' for antiferromagnetic Ising systems in general. A numerical analysis indicates that for the quadratic lattice with $t_1 \uparrow 1$, $t_2 \uparrow 0$ the critical behaviour of the leading-order term of χ' is also similar to that of the internal energy.

Finally, the first-order correction χ_1' on χ_0' is calculated explicitly for the quadratic lattice with $(t_1,t_2) \approx (-1,0)$, and the region of validity of the approximation $\chi' \approx \chi_0'$ is discussed.

§ 1. Introduction

In three preceding papers [1)2)3)], to be referred to as I, II and III henceforth, we have investigated the critical behaviour of the reduced initial susceptibility χ' of ferromagnetic spin-$\frac{1}{2}$ Ising systems on Cartesian lattices in which the coupling constants along d-1 lattice axes are vanishing small in comparison with the coupling constant along the remaining lattice axis. The present paper is also devoted to a study of Ising systems on such extremely anisotropic lattices; however, we now consider cases where the coupling constants J_i along one or more lattice axes are negative. We shall call Ising systems with the latter property <u>antiferromagnetic</u>; this common designation is motivated by the fact that all these systems have the property in common that the net magnetization in the ground state vanishes.

As before we use the high-temperature series expansion for χ' in the variables $t_i = \tanh\beta J_i$, $i = 1,\ldots,d$; we choose J_1 as the "large" coupling constant. The critical behaviour for the afore-said cases of extreme anisotropy occurs near $(t_1,t_2,\ldots,t_d) = (\pm 1,0,\ldots,0)$, the $\pm$ sign corresponding to the sign of J_1. Therefore we rearrange the series into one in the variables $t_2,\ldots,t_d$ with coefficients depending on t_1, as we did in I, II and III.

The first case considered in this paper is that where t_1 lies near -1; the signs of $t_2,\ldots,t_d$ are not specified (§ 2). Oguchi's method for deriving series expansions is found to be even more laborious in this case than in the ferromagnetic cases dealt with before. In view of this fact the expression of Sykes as generalized in III, which is equally valid for negative values of one or more t_i, and which reduces substantially the labour involved in developing series expansions, is still more useful than in the case of ferromagnetic systems. It turns out that for $t_1 \approx -1$ this expression forms even an ideal starting point: from it we can in a straightforward way show that the term χ_0' in χ' which is of leading order in $1+t_1$ for $(t_1,t_2,\ldots,t_d) \to (-1,0,\ldots,0)$, to be called the <u>leading-order term</u> of χ' for this limit, is half the (similarly defined) leading-order term of $1 + \langle\sigma_1\sigma_2\rangle$, where $\langle\sigma_1\sigma_2\rangle$ is the spin-spin correlation function for two nearest neighbours along the lattice axis with coupling constant J_1. It is not clear how this relation could be established on the basis of Oguchi's method. A further analysis of Sykes' expression shows that also the next-leading-order term χ_1' in χ' can be related in a simple way to $\langle\sigma_1\sigma_2\rangle$. Using the explicit formula for $\langle\sigma_1\sigma_2\rangle$ which is available for the quadratic lattice, we are able to derive, via these two relations, closed expressions for χ_0' and χ_1' in the case of the quadratic lattice with $(t_1,t_2) \approx (-1,0)$.

The critical behaviour of the leading-order term χ_0' for the cases with $(t_1,t_2,\dots,t_d) \approx (-1,0,\dots,0)$ as well as for the antiferromagnetic cases with $(t_1,t_2,\dots,t_d) \approx (1,0,\dots,0)$ is investigated in § 3. According to the relation between the leading-order terms of χ' and $1 + \langle\sigma_1\sigma_2\rangle$ derived in § 2, these two quantities have for $(t_1,t_2,\dots,t_d) \to (-1,0,\dots,0)$ a similar critical behaviour. On the basis of this similarity in combination with a hypothesis about the similarity in critical behaviour of thermodynamic quantities and their leading-order terms we show that in all antiferromagnetic cases the susceptibility and the internal energy have the same critical behaviour, which is furthermore independent of the signs of $t_1,\dots,t_d$. This result has been suggested before[4-9] on the basis of theoretical arguments and experimental evidence. Since for the quadratic lattice with $(t_1,t_2) \approx (-1,0)$ χ_0' is known in analytic form, a complete analysis of its critical behaviour is possible in this case. We find that χ_0' displays indeed the well-known critical behaviour of the internal energy of the quadratic lattice; this analysis shows also explicitly that χ_0' yields the correct asymptotic form of the critical equation for this case. Similar conclusions are drawn from a (numerical) study for the quadratic lattice with $t_1 \uparrow 1$, $t_2 \uparrow 0$.

The paper ends with a summary of the main results obtained in this and the preceding papers for Ising systems on extremely anisotropic lattices (§ 4). By a comparison of the expressions derived in this paper for χ_0' and χ_1' for the quadratic lattice with $(t_1,t_2) \approx (-1,0)$, we can draw some conclusions about the region in the t_1,t_2-plane where χ_0' may be considered to be a good approximation to χ'.

§ 2. The susceptibility of the d-dimensional Cartesian lattice with $J_1 < 0$, $|J_i| \ll |J_1|$, $i = 2,\dots,d$

We consider a spin-$\frac{1}{2}$ Ising system on a d-dimensional Cartesian lattice as introduced in I, § 1, for which the coupling constant along one of the lattice axes, J_1, is negative and, in absolute value, much larger than the coupling constants along the other lattice axes, $J_2,\dots,J_d$, which may be positive or negative. The critical behaviour of such a system occurs near $(t_1,t_2,\dots,t_d) = (-1,0,\dots,0)$; therefore, we write, as in the previous papers, the reduced initial susceptibility χ' as a multiple power series in $t_2,\dots,t_d$ with coefficients depending on t_1:

$$\chi' = 1 + \sum_{n_2=0}^{\infty} \cdots \sum_{n_d=0}^{\infty} a_{n_2 \ldots n_d}(t_1) t_2^{n_2} \ldots t_d^{n_d}. \tag{1}$$

In contrast to the ferromagnetic case, we are now interested in the behaviour of the coefficients $a_{n_2 \ldots n_d}(t_1)$ for $t_1 \to -1$. Since the function $(1-t_1)^{-1}$ remains finite for $t_1 \to -1$, one cannot find the leading-order term χ'_0 of χ' for this case simply by restricting oneself to those classes of 2-graphs whose contributions to these coefficients contain as many factors $(1-t_1)^{-1}$ as possible, as we did for ferromagnetic Ising systems (I, II, III). Nevertheless, Oguchi's method for developing series expansions for χ', adapted to extremely anisotropic lattices, on which I and II are founded, still gives an insight into the most relevant properties of the coefficients $a_{n_2 \ldots n_d}(t_1)$.

Let us again consider the division of all 2-graphs belonging to a given class into pieces. The considerations of I, § 4, which remain valid for lattices of arbitrary dimensionality, imply that each piece yields a factor $c_\mu(t_1) \equiv t_1^\mu/(1-t_1^\mu)$, where μ is the multiplicity of the piece, in the contribution from the class in question to the corresponding coefficient; this statement is equally valid for positive and negative values of t_1 (which is, by definition, in absolute value less than 1). For odd values of μ, $c_\mu(t_1)$ remains finite for $t_1 \to -1$; if μ is even, however, $c_\mu(t_1)$ diverges as $(1+t_1)^{-1}$. Therefore, we expect that the number of pieces of <u>even</u> multiplicity in the 2-graphs of a given class rather than the total number of pieces plays a central role for $t_1 \to -1$. As in the ferromagnetic case we may group the contributions to the coefficients $a_{n_2 \ldots n_d}(t_1)$ together according to the way in which they diverge for $t_1 \to -1$; the most divergent contributions will now come from those classes of 2-graphs that contain as many pieces of <u>even</u> multiplicity as possible. We shall say that a piece of even (odd) multiplicity has parity +(-). For given values of $n_2,\ldots,n_d$ the maximal number of pieces of even multiplicity in a 2-graph is $\Sigma_{i=2}^{d} n_i$, which we shall abbreviate by n. For notational convenience we prove this for the case d=2 (i.e. the quadratic lattice), where $n=n_2$.

Consider the infinite y-chains of the lattice which divide the 2-graphs of a given class into pieces. The parities of the multiplicities μ_ℓ and μ_r of two pieces ℓ and r which border on such an infinite y-chain at its left- or right-hand side respectively, are different if and only if this chain goes through exactly one point of odd valence (and an arbitrary number of y-lines) of these 2-graphs. This "parity rule" applies also to the most left (right)-hand infinite y-chain if, for this purpose, one formally treats the empty space lying at its left (right)-hand side as a piece of multiplicity zero and hence of parity +.

For the proof of the parity rule only the properties of the (multi-) graph formed by the points and the lines of a 2-graph, and not its overlapping-structure, are relevant. We consider, for the (multi-) graph G corresponding to a given 2-graph, and for a given infinite y-chain L, the set P of points in G which lie on L. We form the sum V of the valences in G of these points and write it as $V = V_{\ell x} + V_{rx} + V_y$, where $V_{\ell x}$ (V_{rx}) is the number of x-lines of G whose right (left)-hand endpoint is a point of P, and V_y is the number of incidences between a y-line of G and a point of P. Obviously, $V_{\ell x} = \mu_\ell$, $V_{rx} = \mu_r$, and V_y is twice the number of y-lines of G lying on L, and hence even. Consequently, V is even (odd) if $\mu_\ell + \mu_r$ is even (odd), i.e. if the pieces ℓ and r have the same (opposite) parity. On the other hand, V is even (odd) if the number of points of odd valence in P is even (odd). It follows that the parities of the pieces ℓ and r are different if and only if P contains an odd number of points of odd valence. Since in 2-graphs there are only two points of odd valence, the only possibility for a change in parity is that the y-chain goes through exactly <u>one</u> point of odd valence (and an arbitrary number of points of even valence) of G.

In 2-graphs in which the two points of odd valence lie on two different infinite y-chains, L_1 and L_2, one or more pieces lie between these chains. It follows from the foregoing that all pieces not lying between L_1 and L_2, including at least the two empty pieces, have the same parity. This parity is + because the empty pieces have zero, and hence even, multiplicity. The parity of the (one or more) pieces lying between L_1 and L_2 is opposite, and hence −, so that they yield factors $c_\mu(t_1)$ which remain finite for $t_1 \to -1$. Since the maximal number of pieces is n+1, the maximal number of pieces with parity + is n.

On the other hand, if the two points of odd valence lie on one and the same infinite y-chain, all pieces have parity +. However, in this case the maximal number of pieces is n, so that again the maximal number of pieces with parity + is n, which completes our proof for the quadratic lattice. The generalization of this proof to arbitrary values of d is straightforward.

If $c_\mu(t_1)$ is written as a series in ascending powers of $1+t_1$, the dominant term is, for even μ, proportional to $(1+t_1)^{-1}$. Since each piece of even multiplicity yields a factor $c_\mu(t_1)$, and since the maximal number of pieces of parity + in a 2-graph is n, the dominant terms in $a_{n_2 \ldots n_d}(t_1)$ are proportional to $(1+t_1)^{-n}$. Combining this with the factor $t_2^{n_2} \ldots t_d^{n_d}$ multiplying $a_{n_2 \ldots n_d}(t_1)$, one would conclude that the leading-order term χ_L' in χ' for $(t_1, t_2, \ldots, t_d) \to$

$\to (-1,0,\ldots,0)$ is a multiple power series in the variables $t_2/(1+t_1),\ldots,t_d/(1+t_1)$, with no multiplying factor depending on $1+t_1$ alone. A closer investigation shows, however, that the classes of 2-graphs containing the maximal possible number of pieces of even multiplicity can be grouped together in such a way that their dominant contributions cancel completely. We are thereby led to expect that the actual expression of χ_0' in terms of t_1 and t_2 is of the form

$$(1+t_1)^p \sum_{n_2=0}^{\infty} \cdots \sum_{n_d=0}^{\infty} c_{n_2\ldots n_d 0} \left(\frac{t_2}{1+t_1}\right)^{n_2} \cdots \left(\frac{t_d}{1+t_1}\right)^{n_d}, \qquad (2)$$

where the $c_{n_2\ldots n_d 0}$ are constants which are not all zero, and where p is an integer larger than zero. The expression (2) differs markedly from the corresponding expression for ferromagnetic cases in that the power series is multiplied by a factor which vanishes rather than diverges asymptotically.

We shall not prove the above-mentioned cancellation explicitly because we shall show in a more efficient way that χ_0' is of the form (2). We only remark that the result was to be expected for a simple reason. If χ_0' were actually a multiple power series with non-zero coefficients in the variables $t_i/(1+t_1)$, $i=2,\ldots,d$, it would not have a unique limit for $(t_1,t_2,\ldots,t_d) \to (-1,0,\ldots,0)$. For instance, if the point $(-1,0,\ldots,0)$ is approached along a straight line such that the power series converges for the values of the variables $t_i/(1+t_1)$, $i=2,\ldots,d$, corresponding to this straight line, χ_0' would tend to a limiting value depending on these values of the $t_i/(1+t_1)$. In other words, χ_0', and hence χ', would have a finite but infinitely many-valued limit if the point $(-1,0,\ldots,0)$ is approached from the "high-temperature" side, and consequently χ' would not be a well-behaved thermodynamic quantity. Only a cancellation of the dominant terms can remove the many-valuedness, making $\chi'(-1,0,\ldots,0)$ vanish. (In the ferromagnetic case, dealt with in I and II, the many-valuedness of the multiple power series occurring in χ_0' is removed by the multiplying factor $(1-t_1)^{-1}$, which makes χ_0' infinite rather than zero).

In order to find the coefficients $c_{n_2\ldots n_d 0}$ occurring in the expression (2) one has to consider also classes of 2-graphs which do not contain the maximal possible number of pieces with even multiplicity. For p=1 e.g., which will be shown below to be the correct value of p, both the next-dominant contributions from the classes of 2-graphs containing n pieces of even multiplicity and the dominant contributions from the classes with n-1 pieces of even multiplicity have to be taken into account. Such a programme seems to be even more laborious than that followed in I and II for the ferromagnetic case, so that any reduction in the counting procedure is welcome. We therefore return to the generalized formula of Sykes for χ' as derived in III, which proved to be very efficient for the case $t_1 \to +1$.

We recall that for a d-dimensional Cartesian lattice χ' can be written as the sum of four terms (III, eq. (9)):

$$\chi' = A + B + C + D, \tag{3}$$

where

$$A = \frac{(-)^d \prod_{i=1}^{d} (1+t_i)}{D^{(d)}} ; \tag{3a}$$

$$B = -2 \sum_{i=1}^{d} \frac{(1-t_i^2)\left(D_i^{(d)}\right)^2 t_i \frac{\partial}{\partial t_i} \sum_{\vec{p}}' \left(g(\vec{p}) \prod_{j=1}^{d} t_j^{n_j} \right)}{\left(D^{(d)}\right)^2} ; \tag{3b}$$

$$C = \frac{\left(\prod_{i=1}^{d} (1+t_i)^2\right) \sum_{\vec{p}}' \left\{ g(\vec{p}) \sum_{s_1=0}^{2} \cdots \sum_{s_d=0}^{2} \left[p_{s_1 \ldots s_d} \, s(s-2) \prod_{j=1}^{d} t_j^{n_j} \right] \right\}}{\left(D^{(d)}\right)^2} ; \tag{3c}$$

$$D = \frac{2\left(\prod_{i=1}^{d} (1+t_i)^2\right) \sum_{\vec{p}}'' \left\{ g(\vec{p})(s_P - 1)(s_Q - 1) \prod_{j=1}^{d} t_j^{n_j} \right\}}{\left(D^{(d)}\right)^2} ; \tag{3d}$$

In these expressions $D_i^{(d)}$ and $D^{(d)}$ are given by

$$D_i^{(d)} = (-)^{d-1} \prod_{\substack{j=1 \\ j\neq i}}^{d} (1+t_j) \tag{3e}$$

and

$$D^{(d)} = (-)^d \prod_{j=1}^{d} (1+t_j) + \sum_{j=1}^{d} 2t_j D_j^{(d)} , \tag{3f}$$

respectively. The meaning of the various symbols is the same as in III.

We begin the discussion of the eqs. (3) for the case $(t_1, t_2, \ldots, t_d) \to \to (-1, 0, \ldots, 0)$ by noting that $D_i^{(d)}$, $i \neq 1$, vanishes in this limit and that

$D_1^{(d)}$ and $D^{(d)}$ approach a finite value; we have in leading order

$$\left.\begin{aligned} D_1^{(d)} &\simeq (-1)^{d-1} \\ D_i^{(d)} &\simeq (-)^{d-1}(1+t_1), \quad i\neq 1 \\ D^{(d)} &\simeq (-)^d\ 2 \end{aligned}\right\}\left[(t_1,t_2,\dots,t_d) \to (-1,0,\dots,0)\right]. \tag{4}$$

From the fact that each of the quantities A, B, C and D contains one or more factors $1+t_1$ in its numerator, whereas its denominator, containing only $D^{(d)}$ or $\left[D^{(d)}\right]^2$, remains finite in the limit considered, it follows at once that χ_0' is of the form (2) with p larger than zero. To determine the value of p we consider the properties of A, B, C and D in more detail.

Substituting eq. (4) into eq. (3a) we find for the leading-order term A_0 of A:

$$A_0 = \tfrac{1}{2}(1+t_1) \;; \tag{5}$$

this contribution to χ_0' is of the form (2) with p=1.

From the discussion on B given in III we know that for given values of $n_2,\dots,n_d$ the overlappings contributing to the inner sum in eq. (3b) consist of n-1 or fewer pieces; by the absence of points of odd valence all these pieces have even multiplicity, so that they all yield a factor $(1+t_1)^{-1}$ in the contribution to this inner sum. The differentiation with respect to t_1 yields therefore, among others, a term proportional to $(1+t_1)^{-n}$. Using the fact that $D_1^{(d)}$ and $D^{(d)}$ remain finite and that $1-t_1^2$ becomes proportional to $(1+t_1)$ for $(t_1,t_2,\dots,t_d) \to (-1,0,\dots,0)$, we find a contribution to χ' which contains n-1 factors $(1+t_1)^{-1}$. Hence, the term corresponding to i=1 in eq. (3b) yields in leading order also a contribution to χ_0' of the form (2) with p=1. Differentiation of the inner sum with respect to the other t_i and subsequent multiplication by $\left[D_i^{(d)}\right]^2$, which by eq. (4) becomes proportional to $(1+t_1)^2$, yields contributions with n-3 factors $(1+t_1)^{-1}$ at most. Hence the terms with $i\neq 1$ in eq. (3b) yield in leading order contributions to χ_0' of the form (2) with p=3.

As mentioned in III the overlappings contributing to C contain n-2 pieces at most; by their definition they all have even multiplicity. Due to the presence of the factor $\Pi_{i=1}^{d}(1+t_i)^2$, the resulting contributions to χ' contain n-4 factors $(1+t_1)^{-1}$ at most. Hence the leading-order term of C is of the form (2) with p=4.

In the 2-graphs contributing to D there are n-1 pieces at most; some of them may have odd multiplicity. Again, the factor $\Pi_{i=1}^{d}(1+t_i)^2$ reduces the number of factors $(1+t_1)^{-1}$ in the contributions to χ' by two, so that even for a class in which all n-1 pieces have even multiplicity, the contribution to χ' contains only n-3 factors $(1+t_1)^{-1}$. So the leading-order term of D is of the form (2) with p=3.

As mentioned before the leading-order term χ'_0 of χ' is of the form (2) with p equal to <u>or</u> larger than 1. From the fact alone that some of the leading-order terms met in the foregoing analysis are of the form (2) with p=1, one may not conclude that χ'_0 has the same property. A cancellation of the leading-order terms with p=1, i.e. those of A and B, would imply that χ'_0 is of the form (2) with p <u>larger</u> than 1. However, such a cancellation can be excluded beforehand by considering the dependence of these leading-order terms on the variables $t_2,\dots,t_d$. In A_0 these variables occur only to the zeroth power (eq. (5)). On the other hand, since all overlappings contributing to the inner sum of eq. (3b) contain two or more lines with a "small" coupling constant $(J_2,\dots,J_d)$, the leading-order term of B consists of contributions in which one or more of the variables $t_2,\dots,t_d$ occur to at least the second power. Hence the leading-order terms of A and B cannot cancel, which implies in turn that the leading-order term of χ' is of the form (2) with p=1:

$$\chi'_0 = (1+t_1) \sum_{n_2=0}^{\infty} \dots \sum_{n_d=0}^{\infty} c_{n_2\dots n_d 0} \left(\frac{t_2}{1+t_1}\right)^{n_2} \dots \left(\frac{t_d}{1+t_1}\right)^{n_d} , \tag{6}$$

in which the $c_{n_2\dots n_d 0}$ are not all zero.

Summarizing the foregoing analysis we have

$$\chi'_0 = A_0 + B_0 , \tag{7}$$

in which A_0 is given by eq. (5) and B_0 is the leading-order term of B for $(t_1,t_2,\dots,t_d) \to (-1,0,\dots,0)$. In contrast to the ferromagnetic case neither the terms of B with $i\neq 1$ nor the quantity D enter into χ'_0. Furthermore, C is of still less importance than it was in the ferromagnetic case.

Using an alternative form for B which follows from III, eq. (11):

$$B = -2 \sum_{i=1}^{d} \frac{\left(D_i^{(d)}\right)^2 t_i \left(\langle\sigma_{\vec a}\sigma_{\vec a+\vec u_i}\rangle - t_i\right)}{\left(D^{(d)}\right)^2} \tag{8}$$

(for the notation see III), and using eq. (4) for $D_1^{(d)}$ and $D^{(d)}$, we find for B_0:

$$B_0 = \tfrac{1}{2}(\langle\sigma_1\sigma_2\rangle - t_1)_0 \,, \tag{9}$$

in which $\langle\sigma_1\sigma_2\rangle$ is the spin-spin correlation function for two nearest neighbours 1 and 2 along the lattice axis with coupling constant J_1, and in which $(\;)_0$ is the leading-order term of the quantity within the brackets. Observe that $\frac{1}{2}(\langle\sigma_1\sigma_2\rangle - t_1)_0$ is the sum of the next-leading-order terms of $\frac{1}{2}\langle\sigma_1\sigma_2\rangle$ and $-\frac{1}{2}t_1$, the leading-order terms cancelling. Therefore we are not allowed to replace this expression by $\frac{1}{2}\langle\sigma_1\sigma_2\rangle_0 - \frac{1}{2}(t_1)_0$; we may, however, write

$$B_0 = \tfrac{1}{2}(\langle\sigma_1\sigma_2\rangle + 1)_0 - \tfrac{1}{2}(1+t_1). \tag{10}$$

Combining eqs. (5), (7) and (10) we find

$$\chi_0' = \tfrac{1}{2}(1 + \langle\sigma_1\sigma_2\rangle)_0. \tag{11}$$

We postpone a discussion of this relation between the quantities χ' and $\langle\sigma_1\sigma_2\rangle$ and its implications for their singular behaviour till § 3.

For the quadratic lattice a closed form for χ_0' can be easily derived from the well-known expression for the partition function. An explicit derivation of $(1 + \langle\sigma_1\sigma_2\rangle)_0$, analogous to the procedure given in III for the ferromagnetic case, is not necessary if we use the following symmetry property of Ising systems on Cartesian lattices: the partition function of an Ising system on a Cartesian lattice keeps the same value if the coupling constants along one or more lattice axes are reversed in sign. It follows that the spin-spin correlation function for two nearest neighbours along a lattice axis for which the coupling constant is reversed, takes the opposite value. Therefore, the leading-order term in $\langle\sigma_1\sigma_2\rangle - t_1$ for $(t_1,t_2) \to (-1,0)$ is the opposite of that for $(t_1,t_2) \to (1,0)$, with t_1 replaced by $-t_1$. In the derivation of III, eq. (19) the leading-order term of $\langle\sigma_1\sigma_2\rangle - t_1$ (see also III, eq. (11)) for $(t_1,t_2) \to (1,0)$ was found to be

$$(1-t_1)\left\{-\frac{1}{2\pi}\int_{-\pi}^{+\pi} d\omega \, \frac{1-2x\cos\omega}{(1-4x\cos\omega + 4x^2)^{\frac{1}{2}}} + 1\right\}, \qquad x = \frac{t_2}{1-t_1}.$$

Writing the integral in terms of the standard complete elliptic integrals K and E of the 1st and 2nd kind with the argument $2(2x)^{\frac{1}{2}} / (1+2x)$, performing the transformation $\omega \to \psi$ defined by $\sin\omega = (1+2x)\sin\psi / (1+2x\sin^2\psi)$, and integrating E by parts, we obtain this leading-order term in the form

$$(1-t_1)\left[-\frac{2}{\pi}E(2x)+1\right] = (1-t_1)\left\{-\frac{2}{\pi}\int_0^{\pi/2}(1-4x^2\sin^2\psi)^{\frac{1}{2}}\,d\psi + 1\right\}.$$

Combining this result with the symmetry property mentioned above we find

$$(<\sigma_1\sigma_2> - t_1)_0 = (1+t_1)\left[\frac{2}{\pi}E(2y) - 1\right],\ \left[(t_1,t_2)\to(-1,0)\right], \tag{12}$$

where $y = t_2/(1+t_1)$.
Substituting this equation into eq. (11) we derive

$$\chi_0' = (1+t_1)\,\frac{1}{\pi}\,E(2y) = (1+t_1)\,\frac{1}{\pi}\int_0^{\pi/2}(1-4y^2\sin^2\psi)^{\frac{1}{2}}\,d\psi. \tag{13}$$

This formula shows explicitly that for the quadratic lattice χ_0' is of the form

$$\chi_0' = (1+t_1)\sum_{n_2=0}^{\infty} c_{n_2 0}\left(\frac{t_2}{1+t_1}\right)^{n_2}, \tag{14}$$

in which the coefficients $c_{n_2 0}$ do not vanish for all n_2.

The discussion on the various kinds of contributions from A, B, C and D given above enables us to draw one more conclusion in addition to eq. (11), viz. one on the next-leading-order term of χ', which we write as

$$\chi_1' = (1+t_1)^2\sum_{n_2=0}^{\infty}\cdots\sum_{n_d=0}^{\infty} c_{n_2\ldots n_d 1}\left(\frac{t_2}{1+t_1}\right)^{n_2}\cdots\left(\frac{t_d}{1+t_1}\right)^{n_d}, \tag{15}$$

where the $c_{n_2\ldots n_d 1}$ are constants. Similarly to χ_0', χ_1' receives its contributions only from A and the term in B which corresponds to i=1; hence we can write

$$\left[\chi'\right]_1 \equiv \chi_0' + \chi_1' = \left[\frac{(-)^d\prod_{i=1}^{d}(1+t_i)}{D^{(d)}} - \frac{2\left[D_1^{(d)}\right]^2 t_1(<\sigma_1\sigma_2> - t_1)}{\left[D^{(d)}\right]^2}\right]_1, \tag{16}$$

in which $[\ \ ]_1$ means a truncated expansion of the quantity within the brackets containing the leading- and the next-leading-order term. For the quadratic lattice it is possible to derive also a closed expression for χ_1' from the formulae for A and B. Writing A and B in terms of the variables $1+t_1$ and $y = t_2/(1+t_1)$, and expanding them up to second order in $1+t_1$, we find

$$\chi_1' = -\frac{(1+t_1)^2}{2}\left\{\frac{1}{4\pi}\int_{-\pi}^{+\pi} d\omega \frac{1-2y\cos\omega}{(1-4y\cos\omega+4y^2)^{\frac{1}{2}}} + \frac{1}{2\pi}\int_{-\pi}^{+\pi} d\omega \frac{-1+6y\cos\omega-6y^2(1+\cos^2\omega)+8y^3\cos\omega}{(1-4y\cos\omega+4y^2)^{3/2}}\right\},$$

which can be written as

$$\chi_1' = \frac{(1+t_1)^2}{2\pi} K(2y) = \frac{(1+t_1)^2}{2\pi}\int_0^{\pi/2} \frac{d\psi}{(1-4y^2\sin^2\psi)^{\frac{1}{2}}} . \tag{17}$$

§ 3. Critical behaviour of antiferromagnetic Ising systems

We now come to the investigation of the critical behaviour of antiferromagnetic Ising systems, in the sense of one or more J_i being negative, on extremely anisotropic lattices.

Let us first consider the case $J_1<0, |J_i|<<|J_1|$, $i=2,\ldots,d$, treated in § 2. We begin the discussion of the critical behaviour of the susceptibility for this case with an analysis of eq. (6), similar to the analysis given in I, § 6 of the corresponding equation for the ferromagnetic case (and d=2), I, eq. (40). According to eq. (6), χ_0' is the product of a factor $1+t_1$ and a multiple power series in the variables $t_i/(1+t_1)$ $(i=2,\ldots,d)$. It follows that χ_0' behaves as $1+t_1$ if one approaches the point $(-1,0,\ldots,0)$ in the $t_1,\ldots,t_d$-space in such a way that in this limit no singularity arises in χ_0' on the side of the power series, which is achieved e.g. by approaching $(-1,0,\ldots,0)$ along a straight line with the property that the series converges for the corresponding values of the $t_i/(1+t_1)$. Under these conditions the critical behaviour of χ_0' is the same as that of the susceptibility χ' of the antiferromagnetic linear Ising chain:

$$\chi' = \frac{1+t_1}{1-t_1} \sim 1+t_1 \qquad (t_1 \to -1). \tag{18}$$

In this respect the factor $(1+t_1)$ in eq. (6) is completely analogous to the factor $(1-t_1)^{-1}$ in I, eq. (40) (and in II, eq. (5), the extension of I, eq. (40) to d=3), which also describes the pseudo-one-dimensional behaviour, under appropriate conditions, of a more-dimensional system.

On the other hand, we expect that, just as in the cases dealt with in I and II, the d-dimensional nature of the system under consideration is contained in the power series. For the case where all t_i are positive it was found in I (II) for d=2(3) that the singular behaviour in the variables $t_i/(1-t_1)$, i=2(2,3), of the power series occurring in the term in χ' which is of leading order in

$1-t_1$ is, within the limits of accuracy, the same as the (supposedly universal) critical behaviour of the susceptibility itself with respect to Δ_d. Moreover, it turned out that in the case d=2 the series diverges for $x \equiv t_2/(1-t_1)$ satisfying, again within the limits of accuracy, the equation

$$1 - 2x = 0, \tag{19}$$

which is the asymptotic form of the critical equation. Rather than distilling from these indications a hypothesis on the relation between the critical behaviour of χ_0' and χ' for the antiferromagnetic case under consideration only, we turn at once to the most general case of extreme anisotropy where the coupling constant J_1 is (in absolute magnitude) much larger than the other ones, i.e. to arbitrary values of d and arbitrary signs of $t_1,\ldots,t_d$. For each lattice of this type we denote by z_i, $i=2,\ldots,d$, the variables $x_i = t_i/(1-t_1)$ or $y_i = t_i/(1+t_1)$ occurring in the multiple power series in the leading-order term of χ', and by

$$\phi_d(z_2,\ldots,z_d) = 0 \tag{20}$$

the "critical equation" of this series, i.e. the equation which is satisfied by those combinations of values of the variables z_i for which the series becomes singular. We now assume that

a) eq. (20) is the asymptotic form of the critical equation

$$\Delta_d(t_1,\ldots,t_d) = 0 . \tag{21}$$

This assumption formed the basis of our investigations in I and II; as can be seen from I, eq. (39) and II, eq. (4) it amounts (for the ferromagnetic case) to the interchange of the limit $1-t_1 \to 0$ taken in the coefficients of the power series, and the infinite summation(s) in this series.

b) the singular behaviour of the series can be described in terms of the variable $\phi_d(z_2,\ldots,z_d)$ alone. This may be called universality for the leading-order term of χ'; the numerical results for the ferromagnetic simple cubic lattice (II) are consistent with this assumption.

c) this singular behaviour in terms of ϕ_d is of the same form as the critical behaviour of χ' in terms of Δ_d for the given choice of signs of the $t_1,\ldots,t_d$. Although it was not stated explicitly in I and II, this assumption was found to be correct, within the limits of accuracy, for the ferromagnetic quadratic and simple cubic lattice; moreover, we used it for obtaining a better estimate for the region of convergence of the series.

We shall call this threefold assumption, which states that χ' is represented asymptotically by its leading-order term as far as its critical properties are concerned, the <u>leading-order hypothesis</u> for χ'.

Let us now return to the case $(t_1, t_2, \ldots, t_d) \approx (-1, 0, \ldots, 0)$ and consider the implications of the leading-order hypothesis for it. An adequate starting point is the relation (11) between χ_0' and $(1 + \langle\sigma_1\sigma_2\rangle)_0$, because from this relation in combination with the hypothesis we can draw an important conclusion about the critical behaviour of the susceptibility of antiferromagnetic Ising systems in general. According to eq. (11), the singular behaviour of the power series in eq. (6) is the same as that of the power series in the variables $t_i/(1+t_1)$, $i \neq 1$, occurring in $(1 + \langle\sigma_1\sigma_2\rangle)_0$. If we make a leading-order hypothesis for $1 + \langle\sigma_1\sigma_2\rangle$ similar to the one for χ', we arrive at the statement that χ' itself has the same critical behaviour as $1 + \langle\sigma_1\sigma_2\rangle$ (and hence as $\langle\sigma_1\sigma_2\rangle$) for $t_1 < 0$, and arbitrary signs of $t_2, \ldots, t_d$, for which cases eq. (11) was derived. Furthermore, the behaviour of χ_0' and, according to the leading-order hypothesis, of χ' is independent of the signs of $t_2, \ldots, t_d$; for, the symmetry property of Ising systems on Cartesian lattices mentioned in § 2 implies that $\langle\sigma_1\sigma_2\rangle$, and hence $(1 + \langle\sigma_1\sigma_2\rangle)_0$ and χ_0', are even functions of $t_2, \ldots, t_d$, so that the power series in eq. (6) contains only even powers of each variable $t_i/(1+t_1)$, $i = 2, \ldots, d$.

Obviously, χ' is a symmetric function of $t_1, \ldots, t_d$, so that its value is left unchanged under any permutation of these variables. Since any case where t_1 is positive but one t_i, $i \neq 1$, at least is negative, can by a suitable permutation be transformed into a case where t_1 is negative, we conclude that the critical behaviour of χ' is the same for all antiferromagnetic cases. Furthermore, this behaviour is the same as that displayed for $t_1 < 0$ by $\langle\sigma_1\sigma_2\rangle$; on the other hand, $|\langle\sigma_1\sigma_2\rangle|$ does not change if some t_i are reversed in sign, so that we may say that χ' and $\langle\sigma_1\sigma_2\rangle$ have the same critical behaviour for each antiferromagnetic case. Since the x-axis does not play a privileged role in χ', this statement may be generalized to the extent that χ' and <u>all</u> nearest-neighbour spin-spin correlation functions (and hence the internal energy) have the same critical behaviour for all antiferromagnetic cases.

This relation between χ' and the nearest-neighbour spin-spin correlation functions is strongly reminiscent of the results obtained by Fisher et al.[4-8] for the critical behaviour of antiferromagnetic Ising systems on <u>isotropic</u> lattices. In refs. 4 and 5 Fisher gave general arguments why the singularities in these quantities at the critical point should be of the same nature. Furthermore, he was able to calculate exactly the thermodynamic quantities, including χ', of a so-called super-exchange antiferromagnet on a decorated quadratic lattice in an external magnetic field [6]. It turned out that for this model χ' (in zero field) can in a simple way be expressed in terms of the internal energy alone; according to this relation these two quantities have the same critical behaviour. Thirdly, the numerical studies presented in refs. 7 and 8 support the statement that χ' and the nearest-neighbour spin-spin correlation functions have a similar critical behaviour. Finally, there is also some experimental evidence for this similarity [9]. We have now supplemented these data with a simple explicit relation between χ_0' and $(1 + \langle\sigma_1\sigma_2\rangle)_0$, valid for Cartesian lattices of arbitrary dimensionality in the limit of extreme anisotropy, and, through the leading-order hypothesis, with alternative arguments for the relation between the critical behaviour of χ' and the internal energy.

For the quadratic lattice the leading-order hypothesis for χ' can be checked explicitly for the case $(t_1,t_2) \approx (-1,0)$ with the aid of eq. (13). The critical equation for cases where one or both t_i are negative can be found from the one for $t_1,t_2>0$ by using the above-mentioned symmetry property of Ising systems on Cartesian lattices; in this way we find for $t_1<0$, $t_2>0$ the equation

$$1 + t_1 - t_2 + t_1 t_2 = 0, \qquad (22)$$

and for $t_1<0$, $t_2<0$ the equation

$$1 + t_1 + t_2 - t_1 t_2 = 0. \qquad (23)$$

The asymptotic form for $t_2 \to 0$ of these equations is given by

$$\left.\begin{array}{ll} 1 + t_1 \approx \pm 2t_2 , & \begin{pmatrix} t_2 \downarrow 0 \\ t_2 \uparrow 0 \end{pmatrix} \\ \text{or} \quad 1 \mp 2y \approx 0 & \end{array}\right\} . \qquad (24)$$

The integral occurring in the r.h.s. of eq. (13), being the complete elliptic integral E of the second kind, becomes singular for $|y| = \frac{1}{2}$. For the critical behaviour of χ_0' we find

$$\chi_0' = (1+t_1)\,\frac{1}{\pi}\left[1 - \tfrac{1}{4}(1-4y^2)\,\log(1-4y^2) + \ldots\right] \qquad (|2y|\uparrow 1), \qquad (25)$$

which is indeed of the type predicted for χ' in refs. 7 and 8. Hence parts a and c of the leading-order hypothesis for χ' are established to their full extent for the case $(t_1,t_2) \to (-1,0)$, part b is trivial for d=2. Comparing with the well-known critical behaviour of $<\sigma_1\sigma_2>$, and taking into account that eq. (25) has been derived (via eq. (13)) from eq. (11), in which $(1 + <\sigma_1\sigma_2>)_0$ appears, one may look upon eq. (25) also as a confirmation of the leading-order hypothesis for $1 + <\sigma_1\sigma_2>$.

For Cartesian lattices with d>2 no analytic expression for $<\sigma_1\sigma_2>$ is available, so that all our information about χ_0' has to come from series expansions. However, it has been observed that for isotropic lattices the nature of the anti-ferromagnetic critical behaviour of χ' is much harder to determine from a given number of terms in a series expansion than that of the ferromagnetic critical behaviour. On the analogy of this we expect in our case that for an adequate numerical analysis we shall need considerably more coefficients of the series expansion for χ_0' than, e.g. for d=3, the very limited number which can be derived from the expansion of the partition function in III, §3c. The fact that the asymptotic form of the critical equation for d>2 is not known exactly (in contrast with the case d=2) makes this need of more coefficients still more urgent. On the other hand, even if we would derive a large number of coefficients for χ_0' we could not check the leading-order hypothesis for χ' to a satisfactory extent for the very reason that the critical behaviour of χ' itself is inaccurately known. Therefore, we shall not perform any calculations for d>2, and hence leave the question of the validity of the leading-order hypothesis open.

We now pass on to cases of extreme anisotropy where the large variable t_1 lies near +1, but where one or more of the other variables are negative. The procedure for finding the leading-order term χ_0' for these cases is quite the same as the one developed for ferromagnetic cases, because the method followed in I, II and III does not depend on the signs of $t_2,\ldots,t_d$. Hence χ_0' is again given by

$$\chi_0' = \frac{1}{1-t_1} \sum_{n_2=0}^{\infty} \cdots \sum_{n_d=0}^{\infty} b_{n_2\ldots n_d 0} \left(\frac{t_2}{1-t_1}\right)^{n_2} \cdots \left(\frac{t_d}{1-t_1}\right)^{n_d}, \tag{26}$$

which expression was studied in I and II for positive values of $t_2,\ldots,t_d$. We again restrict ourselves to the case d=2, where the critical equation and the critical behaviour of χ' are known. The critical equation of the quadratic lattice for $t_1>0$, $t_2<0$ reads

$$1 - t_1 + t_2 + t_1t_2 = 0, \tag{27}$$

of which the asymptotic form for $(t_1,t_2) \approx (1,0)$ is

$$1 - t_1 \approx -2t_2 \qquad (t_2\uparrow 0). \tag{28}$$

We have seen earlier in this section that in cases where one or more t_i are negative the singular behaviour of χ' should be the same as that of the internal energy. We therefore expect that the power series $\Sigma_{n_2} b_{n_2 0}(t_2/(1-t_1))^{n_2}$, which we denote by B(x) henceforth, with $x=t_2/(1-t_1)$, exhibits in addition to the ferromagnetic singularity for $x\uparrow\frac{1}{2}$, described in I, § 6, the following singular behaviour for $x\downarrow-\frac{1}{2}$:

$$B(x) \simeq B(-\tfrac{1}{2}) - a(1+2x)\log(1+2x) \qquad (x\downarrow-\tfrac{1}{2}), \tag{29}$$

where a is a constant, which in conformity with the terminology of ref. 8 may be called the antiferromagnetic amplitude.

For lack of a relation like eq. (11), which enabled us to find χ_0' for d=2, $(t_1,t_2)\approx(-1,0)$, in closed form, we now depend fully on numerical methods for the investigation of the critical behaviour of χ_0'. As has been pointed out above, an antiferromagnetic singularity, e.g. of the type occurring in eq. (29), is harder to establish from a given number of coefficients of a series than ferromagnetic singularities of the type $(1-bx)^{-p}$. Nevertheless, some techniques have been developed which have given reasonable results [8].

The simplest procedure is to assume that the ferromagnetic singularity occurs through a factor $(1-2x)^{-7/4}$ in B(x), i.e. that B(x) can be written as

$$B(x) = (1-2x)^{-7/4}\,Q(x), \tag{30}$$

where Q(x) is regular for $x=\frac{1}{2}$. Using the expansion obtained for B(x) in III, table II, we find

$$Q(x) \equiv \sum_{n=0}^{\infty} q_n x^n = 2 + x - \frac{3}{4}x^2 + \frac{7}{8}x^3 - \frac{199}{192}x^4 + \frac{469}{384}x^5 - \frac{21421}{13824}x^6 + \frac{177559}{82944}x^7 + \dots \,. \tag{31}$$

Since Q(x) is supposed to contain only the antiferromagnetic singularity (see eq. (29)), the numbers $n(n-1)(-\frac{1}{2})^n q_n$ should tend to a limit for $n\to\infty$; for n=2,...,7 these numbers (up to three decimals) are listed in table I.

TABLE I

n	$n(n-1)(-\frac{1}{2})^n q_n$
2	- 0.375
3	- 0.656
4	- 0.777
5	- 0.763
6	- 0.726
7	- 0.702

There is indeed some evidence for the existence of such a limit; from table I we estimate it to lie near -0.7. The amplitude a, which according to the definition of $Q(x)$, eq. (30), is related to this limit, to be called q, as $a=-2^{-7/4}q$, can then be estimated to be 0.2. In order to gain an insight into the accuracy of this method, we may set up a similar procedure for singularities which are slightly different from the one given in eq. (29), e.g. those in which the quantity $(1+2x)\log(1+2x)$ has been replaced by $(1+2x)^p$, where p is slightly less than 1. We find that such singularities with p lying between 0.9 and 1 are also consistent with the coefficients q_n in eq. (31); however, for p=0.9 the agreement is somewhat worse than for the singularity in eq. (29). A similar comparison for the radius of convergence shows that the consistency for singularities of the form $(1+bx)\log(1+bx)$ does not change significantly if b is varied within the range $1.90<b<2.05$.

A more general method for investigating series having several singularities at their radius of convergence consists in writing the series as a sum of quantities each of which displays a given one of the singularities, and adapting this "Ansatz" to the coefficients of the series which are known. On the analogy of ref. 8 we write $B(x)$ as

$$B(x) = A_1(1-2x)^{-7/4} + A_2(1-2x)^{-3/4} - A_3(1+2x)\log(1+2x), \qquad (32)$$

where A_1, A_2 and A_3 are constants, called again the amplitudes of the respective singularities. By taking sets of three consecutive powers x^m, x^{m+1}, x^{m+2} and performing for such a set the above-mentioned adaptation we find the estimates for A_1, A_2 and A_3 given in table II.

TABLE II

Estimates for A_1, A_2 and A_3 for various values of m

m	A_1	A_2	A_3
3	2.382	- 0.310	0.128
4	2.380	- 0.294	0.141
5	2.381	- 0.302	0.149

The estimates for A_1 lie close to one another; to a somewhat less extent this is also the case for A_2, whereas the estimates for A_3 exhibit a larger relative scattering. On the whole these numerical results are consistent with the singular behaviour of B(x) as expressed in eq. (32). Summarizing, we may say that parts a and c of the leading-order hypothesis are, within the limits of accuracy, confirmed also for the quadratic lattice with $(t_1,t_2) \to (1,0)$.

If we adopt the estimates obtained for m=5 for a final representation of χ_0', we may write

$$\chi_0' = \frac{1}{1-t_1} B(x) = \frac{1}{1-t_1}\left\{2.38(1-2x)^{-7/4} - 0.30(1-2x)^{-3/4} - 0.15(1+2x)\log(1+2x) + \psi(x)\right\}, \tag{33}$$

where $\psi(x)$ is chosen in such a way that the expansion in powers of x of the r.h.s. of eq. (33) coincides with the expansion of χ_0' up to the coefficient b_{70}. We find

$$\psi(x) = - 0.08 + 0.42x + 0.17x^2 - 0.04x^3 + 0.02x^4 . \tag{34}$$

From eqs. (33) and (34) we obtain an estimate also for the value assumed by B(x) in its antiferromagnetic singularity:

$$B(-\tfrac{1}{2}) = 0.29. \tag{35}$$

Thus far we have considered only the leading-order term χ_0'. However, as was shown in § 2, it is possible to derive for the quadratic lattice with $(t_1,t_2) \approx (-1,0)$ also the next-leading-order term χ_1' in closed form (eq. (17)). By the well-known properties of the complete elliptic integral of the first kind K, χ_1' becomes singular for $|y|\uparrow\frac{1}{2}$, just like χ_0', which is consistent with a conjecture made in I, § 6, viz. that all terms in χ' other than χ_0' become singular simultaneously with χ_0'. The critical behaviour of χ_1' is different, however: it diverges logarithmically for $|y|\uparrow\frac{1}{2}$. We shall come back to this property in the next section.

§ 4. Summary and concluding remarks.

The investigations reported on in this and the preceding papers include all cases of extreme anisotropy in d-dimensional Cartesian lattices in which the coupling constants along d-1 lattice axes are (in absolute value) much smaller than the remaining coupling constant. In all these cases the leading-order term χ_0' of the susceptibility χ' turned out to be the product of two factors with the following properties:

i) one factor reflects the fact that the system under consideration can be considered as a set of linear Ising chains which are loosely coupled to one another; this factor has the same critical behaviour as these Ising chains;

ii) the other factor is a multiple power series in d-1 variables; this factor contains the d-dimensional structure of the system. The results obtained for this power series are partly numerical and partly analytical. We give a short survey here.

a) For ferromagnetic Ising systems on the quadratic lattice the first seven coefficients of the series were calculated. By an analysis of these coefficients we found that the radius of convergence of the series gives, within the limits of accuracy, the asymptotic form of the critical equation for this case; furthermore, the singular behaviour turned out to be of the same nature as the singularity for the isotropic quadratic lattice (I, § 6).

b) For ferromagnetic Ising systems on the simple cubic lattice the series was also evaluated, this time up to fifth order in the "small" variables t_2 and t_3. Here, however, the numerical information obtained for the region of convergence of the series cannot be compared with analytic data concerning the critical equation, because such data are not available. A comparison for the singular behaviour as under a) is possible; within the limits of accuracy this behaviour was found to be similar to that of the ferromagnetic susceptibility of the isotropic simple cubic lattice (II, § 3).

c) For antiferromagnetic Ising systems for which the large variable t_i lies near -1, we were able to show that χ_0' equals $\frac{1}{2}(1 + \langle\sigma_{\vec{a}}\sigma_{\vec{a}+\vec{u}_i}\rangle)_0$, where $\langle\sigma_{\vec{a}}\sigma_{\vec{a}+\vec{u}_i}\rangle$ is the spin-spin correlation function for nearest neighbours along the i^{th} lattice axis. As a consequence we could obtain a closed expression for χ_0' in the case of the quadratic lattice (and even one for the next-leading-order term χ_1') (this paper, § 2). With the aid of this expression we could, for this case, show rigorously both that the power series yields the asymptotic form of the critical equation, and that its singularity is of the same nature as that displayed by antiferromagnetic Ising systems on the isotropic quadratic lattice (this paper, § 3).

d) The antiferromagnetic quadratic lattice in which the large variable t_i lies near +1, was investigated numerically; in this case the series is the same as the one for ferromagnetic Ising systems. It turned out again that the series yields, within the limits of accuracy, the asymptotic form of the critical equation, the singularity being of the same type as the singularity met in the case mentioned under c) (this paper, § 3).

Both the numerical and the analytical results mentioned here support the leading-order hypothesis, formulated in § 3, which states that the power series in χ_0' describes the critical aspects of the susceptibility itself in the limit of extreme anisotropy. Moreover we were, via this hypothesis, led to a relation between the antiferromagnetic susceptibility and the internal energy which has been generally accepted on other grounds.

Up to now we have paid attention only to the mathematical properties of χ_0'; we now come to the question to what extent χ_0' may be considered as a "physical" quantity, i.e. as a numerically good approximation to χ'. The shortcomings in the latter respect were, for the ferromagnetic quadratic lattice, mentioned in I, § 6. For that case we also conjectured that, broadly speaking, χ_0' would be a better approximation to χ' as the large variable would come closer to 1. For the quadratic lattice with $(t_1,t_2) \rightleftarrows (-1,0)$ we are in a position to make such a statement somewhat more precise owing to the availability of a closed expression for the next-leading-order term χ_1' (eq. (17)). As mentioned at the end of § 3, χ_1' diverges as $\log(1-2|y|)$ for $|y| = |t_2/(1+t_1)| \uparrow \frac{1}{2}$. From this critical behaviour, in combination with the fact that χ_1' by definition contains one factor $1+t_1$ more than χ_0', we expect that a necessary condition on the values of t_1 and t_2 for which χ_1' may be neglected with respect to χ_0', is

$$(1+t_1)\left|\log\left(1 \mp \frac{2t_2}{1+t_1}\right)\right| \ll 1 \qquad (t_2 \gtrless 0), \tag{36}$$

which forms a more precise form of the analogue for the antiferromagnetic case of the above-mentioned conjecture.

The fact that χ_1', in contrast with χ_0', diverges for $|y| \uparrow \frac{1}{2}$, was to be expected on the basis of the hypothesis made in I, § 1, that the critical behaviour of thermodynamic quantities can be expressed in terms of the function $\Delta_d(t_1,\ldots,t_d)$ alone. Combining this hypothesis, applied to χ' for the quadratic lattice, with the conclusion of § 3 that χ' and the internal energy have the same critical behaviour if t_1 and/or t_2 is negative, we can write

$$\chi'(t_1,t_2) \approx \chi'(t_{1c},t_{2c}) + a(t_{1c},t_{2c})\Delta_2(t_1,t_2)\log\Delta_2(t_1,t_2) \qquad \left[(t_1,t_2) \to (t_{1c},t_{2c})\right], \tag{37}$$

where $\Delta_2(t_1,t_2)$ is the l.h.s. of eq. (22) or (23) for the case $t_1<0, t_2>0$ and

$t_1<0$, $t_2<0$, respectively, and where (t_{1c},t_{2c}) is an arbitrary point with $t_{1c},t_{2c} \neq 0$, satisfying $\Delta_2(t_{1c},t_{2c}) = 0$.

Let us first consider the former case. We choose a point (t_{1c},t_{2c}) with the properties just mentioned; by splitting off a factor $1+t_1$ from Δ_2 we can rewrite eq. (37) as

$$\chi'(t_1,t_2) \simeq \chi'(t_{1c},t_{2c}) + a(t_{1c},t_{2c})(1+t_1)\,[1-(1-t_1)y]\,\log(1+t_1) + $$
$$+ a(t_{1c},t_{2c})(1+t_1)\,[1-(1-t_1)y]\,\log[1-(1-t_1)y] \quad \Big[(t_1,t_2)\to(t_{1c},t_{2c}) \neq (-1,0)\Big]. \tag{38}$$

Obviously, the singular behaviour of χ' is contained in the third term of the r.h.s. of this equation; we denote it by χ'_s and rewrite it as a power series in $1+t_1$ with coefficients ψ_n depending on y, which yields after a simple calculation:

$$\chi'_s(t_1,t_2) = a(t_{1c},t_{2c})(1+t_1)\sum_{n=0}^{\infty}\psi_n(y)(1+t_1)^n,$$

$$\left.\begin{aligned} &\text{with } \psi_0(y) = (1-2y)\log(1-2y),\\ &\psi_1(y) = y\log(1-2y) + y,\\ &\psi_n(y) = \frac{(-y)^n}{n(n-1)(1-2y)^{n-1}},\ n \geq 2\,. \end{aligned}\right\} \tag{39}$$

This series for the "model function" χ'_s displays all features of the series for χ' in ascending powers of $1\pm t_1$ discussed in this and the preceding papers. First, the coefficient $\psi_0(y)$ in the term of leading order in $1+t_1$ becomes singular for $y=\frac{1}{2}$, i.e. for y satisfying the asymptotic form of the critical equation for this case, and its critical behaviour in terms of $\phi_2=1-2y$ is the same as that of χ'_s in terms of $1-(1-t_1)y$; moreover, it is the same as that of χ'_0 and χ'. Secondly, the coefficient $\psi_1(y)$ of the term which is of next-leading order in $1+t_1$ diverges for $y\uparrow\frac{1}{2}$; its singularity is the same as that of χ'_1. A new feature is that the critical behaviour of the coefficients of the higher powers of $1+t_1$ is also known: they diverge stronger and stronger as n increases. We see that in spite of the fact that ψ_1, and even all ψ_n for $n\geq 1$ diverge for $y\uparrow\frac{1}{2}$, and that consequently the series (39) ceases to make sense for $\frac{1}{2}\leq y<(1-t_1)^{-1}$, where χ'_s itself is still regular, the leading-order term of χ'_s is a reliable tool for studying the asymptotic form of the critical equation.

The fact that the series in eq. (39) starts by a term containing precisely the first power of $1+t_1$ cannot be used to draw conclusions about the critical behaviour of χ' in the limit $t_1\to -1$, because eq. (37) describes only the critical behaviour of χ' as a point $(t_{1c},t_{2c}) \neq (-1,0)$ is approached. To extract from the latter behaviour information on the former, one should know the properties of $a(t_{1c},t_{2c}) = a\left[t_{1c},(1+t_{1c})/(1-t_{1c})\right]$ in the limit $t_{1c} \to -1$.

If the similarity of χ' to χ_s' continues to hold for the higher-order terms, the divergence of the $\psi_n(y)$ as $(1-2y)^{-(n-1)}$ for $n\geq 2$ implies a similar divergence for the terms in χ' which are of order $(1+t_1)^{n+1}$ with $n\geq 2$. An analysis analogous to the one given above, with 1-2y replaced by 1+2y, may be given for the case $t_1<0$, $t_2<0$. On account of the singularity $(1-2|y|)^{-(n-1)}$ in the term of χ' proportional to $(1+t_1)^{n+1}$, $n\geq 2$, thus conjectured, one might expect that the region where χ_0' is a good approximation to χ', is determined by a somewhat sharper condition than eq. (36); instead, it would be given by

$$1 + t_1 \ll 1 \mp \frac{2t_2}{1+t_1} \qquad (t_2 \gtrless 0). \tag{40}$$

The derivation of eq. (39) depends only on the hypothesis that critical behaviour can be described in terms of Δ_d alone, which we enunciated in I, § 1 as a property of Ising systems on lattices of arbitrary dimensionality, either ferro- or antiferromagnetic. We expect therefore that a procedure similar to the one presented here, is possible for any case of extreme anisotropy $(t_1,t_2,\ldots,t_d) \to (\pm 1,0,\ldots,0)$. This would imply that the successive terms $\chi_0', \chi_1',\ldots, \chi_n',\ldots$ in χ', considered as functions of the variables $t_i/(1\pm t_1)$, $i\neq 1$, diverge with a critical index of the form n+p, where p is a constant; in other words, the divergence of the terms χ_n' would be stronger as n increases. As a consequence one might derive for each case of extreme anisotropy conditions similar to eq. (40) on the region in the $t_1,\ldots,t_d$-space where χ_0' is a good approximation to χ'; e.g. for the ferromagnetic quadratic lattice with $(t_1,t_2)\approx(1,0)$ the region would be given by

$$1 - t_1 \ll 1 - \frac{2t_2}{1-t_1}. \tag{41}$$

Considering again χ_0' as a tool for investigating the critical equation without any questioning about its direct physical meaning, we may say that the results mentioned above under a) through d) add to our confidence in the relevance of the numerical results obtained in II to the asymptotic form of the critical equation of the simple cubic lattice. In the following and last paper of this series we shall continue the study of this equation by considering cases of extreme anisotropy in which only one coupling constant is very small with respect to the other coupling constants.

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CRITICAL BEHAVIOUR OF ISING SYSTEMS ON EXTREMELY ANISOTROPIC LATTICES

V. Some aspects of the crossover from d to d-1 dimensions

Synopsis

The critical equation of a ferromagnetic spin-$\frac{1}{2}$ Ising system on a d-dimensional Cartesian lattice with coupling constants $J_1,\ldots,J_d$ along the d lattice axes is investigated by means of a series expansion for the reduced initial susceptibility χ_d' in the variable $t_d = \tanh\beta J_d$. The coefficients a_n in this series are sums of products of multiple-spin correlation functions on a (d-1)-dimensional Cartesian lattice with coupling constants $J_1,\ldots,J_{d-1}$. It is shown that

$$a_0 = \chi'_{d-1}\ ,\quad a_1 = 2\chi'^{\,2}_{d-1}\ ,\quad 2\chi'^{\,3}_{d-1} \le a_2 \le 4\chi'^{\,3}_{d-1},\quad 0 \le a_3 \le 8\chi'^{\,4}_{d-1}\ ,$$ where

χ'_{d-1} is the susceptibility of the Ising system on the (d-1)-dimensional lattice; for the critical exponent $\gamma^{(n)}$ of a_n $(n\le 3)$ this implies: $\gamma^{(0)}=\gamma_{d-1}$, $\gamma^{(1)}=2\gamma_{d-1}$, $\gamma^{(2)}=3\gamma_{d-1}$, $\gamma^{(3)}\le 4\gamma_{d-1}$, where γ_{d-1} is the critical exponent of χ'_{d-1}. On the basis of the conjecture that $\gamma^{(n)}=(n+1)\gamma_{d-1}$ and a symmetry argument the equation

$$(1-2x_2)^{7/4} + (1-2x_3)^{7/4} = 1 \qquad \left(x_{2(3)} = \frac{\tanh\beta J_{2(3)}}{1-\tanh\beta J_1} \right)$$

is proposed for the asymptotic form of the critical equation of the simple cubic lattice in the limit $J_2/J_1 \to 0$, $J_3/J_1 \to 0$; this equation is in full agreement with numerical results obtained in a previous paper (II).

§ 1. Introduction

In the preceding papers in this series [1-4], to be referred to as I, II, III and IV henceforth, we have investigated the critical behaviour of the initial susceptibility of spin-$\frac{1}{2}$ Ising systems on Cartesian lattices in which the coupling constant along one lattice axis is (in absolute value) much larger than those along the remaining lattice axes. In the present paper we study the susceptibility and, by means of this quantity, the critical equation for d-dimensional Cartesian lattices in which the coupling constant along one lattice axis, say J_d, is much smaller than the other coupling constants, $J_1,\ldots,J_{d-1}$. A lattice of this type may be considered to consist of (d-1)-dimensional lattices which are weakly coupled to each other. From this point of view it is natural to develop the susceptibility χ'_d *) of the d-dimensional lattice into a power series $\Sigma_n a_n t_d^n$ in the single variable $t_d = \tanh\beta J_d$, in which the coefficients a_n depend on $J_1,\ldots,J_{d-1}$. It appears that the a_n can be expressed completely in terms of multiple-spin correlation functions of the (d-1)-dimensional lattice with coupling constants $J_1,\ldots,J_{d-1}$. For the investigation of the a_n we restrict ourselves to ferromagnetic Ising systems, for which we have the Griffiths inequalities [5] at our disposal. Using these inequalities we can for a_2 and a_3 derive upper and lower bounds in terms of the susceptibility of the (d-1)-dimensional lattice, χ'_{d-1}; a_0 and a_1 are related in a simple way to χ'_{d-1}. As a result we can prove, for $n \leq 2$, that the a_n have the same critical behaviour as $\chi'^{\,n+1}_{d-1}$, and, for n=3, that a_n does not diverge faster than $\chi'^{\,n+1}_{d-1}$.

In conformity with the hypothesis made in I, § 1, that the critical behaviour of thermodynamic quantities can be described in terms of one single function Δ_d of the variables $t_i = \tanh\beta J_i$ (i=1,...,d), we assume that the critical behaviour of χ'_{d-1} is given by

$$\chi'_{d-1} \sim \Delta_{d-1}^{-\gamma_{d-1}} \qquad \left[(t_1,\ldots,t_{d-1}) \to (t_{1c},\ldots,t_{d-1\,c})\right] , \tag{1}$$

where $(t_{1c},\ldots,t_{d-1\,c})$ is an arbitrary point with $t_{1c},\ldots,t_{d-1\,c} > 0$ of the critical hypersurface for the (d-1)-dimensional lattice, and the limit is taken from the high-temperature side. Similarly, we assume that the a_n, being composed of correlation functions, have the critical behaviour

*) In this paper it is desirable to distinguish the susceptibilities of lattices of different dimensionality by an index. This index is not to be confused with the indices 0 and 1 used in the preceding papers to denote the leading-order and next-leading-order terms in the susceptibility.

$$a_n \sim \Delta_{d-1}^{-\gamma^{(n)}} \qquad \left[(t_1,\ldots,t_{d-1}) \to (t_{1c},\ldots,t_{d-1\,c})\right] . \tag{2}$$

By its definition the function $\Delta_{d-1} = \Delta_{d-1}(t_1,\ldots,t_{d-1})$ has the property that the equation

$$\Delta_{d-1}(t_1,\ldots,t_{d-1}) = 0 \tag{3}$$

is the critical equation of the (d-1)-dimensional lattice. In order to make the connection with the standard definition of critical exponents we require that the function Δ_{d-1}, which is not uniquely defined by eq. (3), has the property

$$\Delta_{d-1}\left(\tanh\frac{J_1}{kT},\ldots,\tanh\frac{J_{d-1}}{kT}\right) \sim T - T_c \qquad (T \to T_c), \tag{4}$$

where T_c is the critical temperature of the (d-1)-dimensional lattice with coupling constants $J_1,\ldots,J_{d-1}$.

The results on a_n $(n \leq 3)$ mentioned above imply that the critical exponents $\gamma^{(n)}$ $(n \leq 3)$ satisfy the following relations:

$$\gamma^{(0)} = \gamma_{d-1} , \tag{5}$$

$$\gamma^{(1)} = 2\gamma_{d-1} , \tag{6}$$

$$\gamma^{(2)} = 3\gamma_{d-1} , \tag{7}$$

$$\gamma^{(3)} \leq 4\gamma_{d-1} . \tag{8}$$

A preliminary account of these results was given in a letter [6)*)], in which they were compared with recent work by Liu and Stanley [7)]. Working independently along the same lines, these authors had (for the case $J_1=\ldots=J_{d-1}$) derived the relations (5) and (6), together with the inequalities

$$\gamma^{(2)} \geq 3\gamma_{d-1} , \tag{9}$$

$$\gamma^{(3)} \geq 4\gamma_{d-1} . \tag{10}$$

In their letter they gave a proof of eq. (9), deferring to a later publication the proof of eq. (10), which we have not derived, and which, in combination with the inequality (8), yields the identity

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*) In this letter $\gamma^{(n)}$ is denoted by γ_n, γ_{d-1} by γ, and the coefficients a_n and related quantities by c_n etc.

$$\gamma^{(3)} = 4\gamma_{d-1}. \tag{11}$$

In a subsequent letter [8] Liu and Stanley derived the upper bound for a_2 which is implicit in our derivation of eq. (7), and mentioned the analogous upper bound for a_3. Finally, Liu has shown [9] that

$$\gamma^{(4)} \geq 5\gamma_{d-1}. \tag{12}$$

In view of the relations (5), (6), (7) and (11) we make the conjecture that the relation

$$\gamma^{(n)} = (n+1)\gamma_{d-1} \tag{13}$$

is true for all n. The conjecture is confirmed for d=2, where it is a direct consequence of I, eq. (38). A strong support for its validity for general values of d is derived from the fact that for $n \geq 1$ each a_n contains a term $2\chi'^{\,n+1}_{d-1}$. Evidently, this term is for $n \leq 3$ neither overgrown nor cancelled by the other terms present in a_n; there is no special reason to expect the situation to be different for higher values of n. Furthermore, the numerical results for the s.c. and the f.c.c. lattice, derived from series expansions by Krasnow et al. [10], confirm eq. (13) for $n \leq 5$ within the limits of accuracy. Finally, eq. (13) has been shown to follow from scaling theory arguments [10]. The scaling hypothesis for the free energy with respect to the parameter J_d (or rather to a parameter indicating the "strength" of J_d) leads to the relation

$$\gamma^{(n)} = \gamma_{d-1} + n\phi, \tag{14}$$

where the <u>crossover exponent</u> ϕ is independent of n. Combination of eq. (14) with one of the equations (6), (7) or (11) yields $\phi = \gamma_{d-1}$. This result can also be derived by an independent scaling approach due to Abe [11].

Starting from eq. (2) together with eq. (13) we investigate the way in which the critical equation of the s.c. lattice,

$$\Delta_3(t_1,t_2,t_3) = 0, \tag{15}$$

reduces to that of the quadratic lattice,

$$\Delta_2(t_1,t_2) \equiv 1 - t_1 - t_2 - t_1t_2 = 0, \tag{16}$$

as J_3, and hence t_3, tends to zero. From the analysis of the structure of the series $\Sigma_n a_n t_3^n$ for χ'_3 we find that the critical surface representing eq. (15)

in the t_1,t_2,t_3-space touches the t_1,t_2-plane, and hence, by symmetry, the t_1,t_3- and the t_2,t_3-plane. It follows that in the points (1,0,0), (0,1,0) and (0,0,1) the critical surface has more than one tangent plane; this feature is in agreement with the conical behaviour of the critical surface near these points predicted in II on the basis of a numerical analysis of χ_3 in the case where J_2 and J_3 are much smaller than J_1.

For the precise form of the critical surface in the immediate neighbourhood of the point (1,0,0) we finally propose an explicit formula, based on the behaviour of the surface near the coordinate planes and a symmetry argument. This formula is in full agreement with the numerical results referred to above.

The plan of the paper is as follows. In § 2 we discuss the structure of the coefficients a_n in the power series $\Sigma_n a_n t_d^n$ for χ'_d. In § 3 we derive the relations between a_0,a_1,a_2 and a_3 on the one hand, and χ'_{d-1} on the other hand, and discuss their implications for the critical behaviour of these coefficients. Section 4 is devoted to the investigation of the critical surface of the s.c. lattice.

§ 2. <u>The susceptibility of the d-dimensional Cartesian lattice with $|J_d| \ll |J_i|$, $i=1,\ldots,d-1$</u>

We consider a spin-$\frac{1}{2}$ Ising system on a d-dimensional Cartesian lattice L in which the coupling constant along the d^{th} lattice axis, J_d, is (in absolute value) much smaller than the coupling constants along the remaining d-1 lattice axes, $J_1,\ldots,J_{d-1}$. For such a system the interaction between spins on neighbouring points with the same value of the d^{th} coordinate is relatively strong in comparison with the interaction between spins on points with different values of the d^{th} coordinate. Henceforth we shall call a set of points with a common value of the d^{th} coordinate a <u>layer</u>; these layers are weakly coupled to each other. In conformity with this way of considering the lattice we denote the set of the d coordinates of a point by two indices, one of which is a Latin index ($k, \ell, \ldots$) labelling the points within a layer, whereas the other one is a Greek index ($\kappa,\lambda,\ldots = 1,\ldots,N_d$) representing the d^{th} coordinate; we label the consecutive layers by means of this Greek index as L'_κ. Furthermore, we split the Hamiltonian $\mathcal{H}$ of an Ising system on a finite $N_1\times\ldots\times N_d$ lattice L_N ($N=N_1N_2\ldots N_d$) with periodic boundary conditions into two parts:

$$\mathcal{H} = \mathcal{H}_0 - J_d \sum_{k,\kappa} \sigma_{k\kappa}\,\sigma_{k\,\kappa+1} \tag{17}$$

with

$$\mathcal{H}_0 = \sum_{\kappa=1}^{N_d} \mathcal{H}'_\kappa \; ; \tag{18}$$

here $\mathcal{H}'_\kappa$ contains the interaction energy for the pairs of nearest neighbours within the layer $L'_{\kappa,N'}$ $(N' = N_1 N_2 \ldots N_{d-1})$. In other words, $\mathcal{H}_0$ contains the interaction energy for pairs of nearest neighbours within the layers, whereas the second term in eq. (17) contains the interaction energy between the layers.

In view of the fact that $|J_d|$ is much smaller than J_i $(i=1,\ldots,d-1)$, we shall expand the reduced initial susceptibility per point of L_N, χ'_d, into a power series in the variable $t_d = \tanh\beta J_d$ with coefficients a_n depending on $J_1,\ldots,J_{d-1}$:

$$\chi'_d = \sum_{n=0}^{\infty} a_n t_d^n \, . \tag{19}$$

For notational convenience we do not indicate explicitly the dependence of the a_n on these coupling constants; furthermore, we do not, as in I, § 3, indicate the dependence of the various quantities occurring in this section, on the numbers $N_1,\ldots,N_d$. In order to derive the series (19), and to find expressions for the coefficients a_n, we start from I, eq. (18), which is valid not only for the infinite lattice L, but also for the finite lattice L_N, and which reads in our notation

$$\chi'_d = \sum_{j,\omega} \langle \sigma_{1\alpha} \sigma_{j\omega} \rangle \, , \tag{20}$$

where $\langle\ \rangle$ denotes a thermal average on L_N; the index α will be reserved here and in the following for the layer containing the fixed point, which is therefore denoted by 1α instead of by 1 as in the preceding papers. Combining eqs. (17) and (20), we have

$$\chi'_d = \sum_{j,\omega} \frac{\Sigma_{\{\sigma\}}\left[\sigma_{1\alpha}\sigma_{j\omega} \exp(-\beta\mathcal{H}_0+\beta J_d \Sigma_{k,\kappa} \sigma_{k\kappa}\sigma_{k\,\kappa+1})\right]}{\Sigma_{\{\sigma\}}\left[\exp(-\beta\mathcal{H}_0+\beta J_d \Sigma_{k,\kappa} \sigma_{k\kappa}\sigma_{k\,\kappa+1})\right]} \, , \tag{21}$$

where the sums $\Sigma_{\{\sigma\}}$ run over all 2^N spin configurations of the Ising system on L_N. Applying the identity

$$\exp(\beta J_d \sigma_{k\kappa}\sigma_{k\,\kappa+1}) = (\cosh\beta J_d)(1 + \sigma_{k\kappa}\sigma_{k\,\kappa+1} \tanh\beta J_d) \tag{22}$$

we can expand the numerator and the denominator of the r.h.s. of eq. (21) into polynomials of degree N_d in t_d:

$$\chi'_d = \sum_{j,\omega} \frac{\sum_{\{\sigma\}}\left[\sigma_{1\alpha}\sigma_{j\omega}\left(1+t_d\sum_{k,\kappa}\sigma_{k\kappa}\sigma_{k\,\kappa+1}+t_d^2\sum'_{k,\kappa;\ell,\lambda}\sigma_{k\kappa}\sigma_{k\,\kappa+1}\sigma_{\ell\lambda}\sigma_{\ell\,\lambda+1}+\ldots\right)\exp(-\beta\mathcal{H}_0)\right]}{\sum_{\{\sigma\}}\left[\left(1+t_d\sum_{k,\kappa}\sigma_{k\kappa}\sigma_{k\,\kappa+1}+t_d^2\sum'_{k,\kappa;\ell,\lambda}\sigma_{k\kappa}\sigma_{k\kappa+1}\sigma_{\ell\lambda}\sigma_{\ell\,\lambda+1}+\ldots\right)\exp(-\beta\mathcal{H}_0)\right]}; \tag{23}$$

here $\Sigma'_{k,\kappa;\ell,\lambda}$ denotes a summation over couples of points $k\kappa$, $\ell\lambda$ with the restrictions that each couple occurs only once, and that $k\kappa$ is different from $\ell\lambda$. Dividing the numerator and the denominator of the r.h.s. of this equation by the partition function $\Sigma_{\{\sigma\}}\exp(-\beta\mathcal{H}_0)$ of the Ising system with Hamiltonian $\mathcal{H}_0$ We obtain

$$\chi'_d = \sum_{j,\omega} \frac{\langle\sigma_{1\alpha}\sigma_{j\omega}\rangle_0+t_d\sum_{k,\kappa}\langle\sigma_{1\alpha}\sigma_{j\omega}\sigma_{k\kappa}\sigma_{k\,\kappa+1}\rangle_0+t_d^2\sum'_{k,\kappa;\ell,\lambda}\langle\sigma_{1\alpha}\sigma_{j\omega}\sigma_{k\kappa}\sigma_{k\,\kappa+1}\sigma_{\ell\lambda}\sigma_{\ell\,\lambda+1}\rangle_0+\ldots}{1+t_d\sum_{k,\kappa}\langle\sigma_{k\kappa}\sigma_{k\,\kappa+1}\rangle_0+t_d^2\sum'_{k,\kappa;\ell,\lambda}\langle\sigma_{k\kappa}\sigma_{k\,\kappa+1}\sigma_{\ell\lambda}\sigma_{\ell\,\lambda+1}\rangle_0+\ldots} \tag{24}$$

where $\langle\ \rangle_0$ represents a thermal average in the Ising system on L_N with Hamiltonian $\mathcal{H}_0$. Since the $\mathcal{H}'_\kappa$ in eq. (18) have no spin variables in common, the thermal average $\langle\ \rangle_0$ of any product of spins can be written as a product of thermal averages $\langle\ \rangle^{(\kappa)}$, $\langle\ \rangle^{(\lambda)}$,... of the products of spins within the layers $L'_{\kappa,N'}$, $L'_{\lambda,N'}$, ..., taken with respect to the layer Hamiltonians $\mathcal{H}'_\kappa$, $\mathcal{H}'_\lambda$,..., respectively. Since all layers are, apart from their label, identical, we may in these averages omit all reference to these labels, and write:

$$\langle\prod_i\sigma_{k_i\kappa}\prod_j\sigma_{\ell_j\lambda}\ldots\rangle_0 = \langle\prod_i\sigma_{k_i\kappa}\rangle^{(\kappa)}\langle\prod_j\sigma_{\ell_j\lambda}\rangle^{(\lambda)}\ldots = \langle\prod_i\sigma_{k_i}\rangle\langle\prod_j\sigma_{\ell_j}\rangle\ldots, \tag{25}$$

where an average $\langle\ \rangle$ including spins with only a Latin index is a thermal average with respect to the Hamiltonian of a (d-1)-dimensional lattice $L'_{N'}$, with coupling constants $J_1,\ldots,J_{d-1}$. Since we consider Ising systems at temperatures above the critical temperature of L - which, as is well known, does not lie below that of L' - we may put all (zero-field) multiple-spin correlation functions containing an odd number of spins equal to zero. It follows that the only non-zero terms in eq. (24) are those in which each layer index occurs an even number (possibly zero) of times. This fact reduces the number of terms which actually have to be taken into account substantially; for instance, in the denominator only the terms with even powers of t_d yield non-vanishing contributions, which corresponds to the well-known fact that the partition function of an Ising system on a Cartesian lattice is even in each t_i. In other words, all terms with odd powers of t_d in the denominator may be omitted right from the beginning.

The further development of χ'_d consists in carrying out the division in eq. (24), which yields a power series of the form (19), in which the coefficients a_n depend, through $\mathcal{H}_0$, on the coupling constants $J_1,\ldots,J_{d-1}$. We investigate the structure of the first few coefficients.

Obviously, the coefficient a_0 is the susceptibility χ'_{d-1} per point of a (d-1)-dimensional lattice $L'_{N'}$. This follows indeed from eq. (24); putting t_d equal to zero in the r.h.s. of this equation we find

$$a_0 = \sum_{j,\omega} \langle\sigma_{1\alpha}\sigma_{j\omega}\rangle_0 \,. \tag{26}$$

Since the only non-zero terms in this sum are those with $\omega=\alpha$, this reduces to

$$a_0 = \sum_{j} \langle\sigma_1\sigma_j\rangle \,, \tag{27}$$

where j runs over all points of $L'_{N'}$; the r.h.s. of this equation is just the susceptibility χ'_{d-1}.

The coefficient a_1 is from eq. (24) found to be

$$a_1 = \sum_{j,\omega}\sum_{k,\kappa} \langle\sigma_{1\alpha}\sigma_{j\omega}\sigma_{k\kappa}\sigma_{k\,\kappa+1}\rangle_0 \,. \tag{28}$$

The only non-zero terms in this expression are those with $\omega=\alpha-1$, $\kappa=\alpha-1$, or $\omega=\alpha+1$, $\kappa=\alpha$, so that we have

$$a_1 = \sum_{j,k} \langle\sigma_{1\alpha}\sigma_{j\,\alpha-1}\sigma_{k\,\alpha-1}\sigma_{k\alpha}\rangle_0 + \sum_{j,k} \langle\sigma_{1\alpha}\sigma_{j\,\alpha+1}\sigma_{k\alpha}\sigma_{k\,\alpha+1}\rangle_0 \,,$$

which by the factorization of thermal averages $\langle\ \rangle_0$, eq. (25), can be written as

$$a_1 = 2\sum_{j,k} \langle\sigma_1\sigma_k\rangle\langle\sigma_k\sigma_j\rangle \,. \tag{29}$$

The coefficient a_2 is given by

$$a_2 = \sum_{j,\omega} \sum_{k,\kappa;\ell,\lambda}{}' <\sigma_{1\alpha}\sigma_{j\omega}\sigma_{k\kappa}\sigma_{k\,\kappa+1}\sigma_{\ell\lambda}\sigma_{\ell\,\lambda+1}>_0 -$$

$$- \sum_{j,\omega} \sum_{k,\kappa;\ell,\lambda}{}' <\sigma_{1\alpha}\sigma_{j\omega}>_0 <\sigma_{k\kappa}\sigma_{k\,\kappa+1}\sigma_{\ell\lambda}\sigma_{\ell\,\lambda+1}>_0 . \tag{30}$$

The first term in this expression yields non-zero contributions in the following three cases:

a) $\omega = \alpha+2, \quad \kappa = \alpha, \quad \lambda = \alpha+1 \quad (\text{or } \lambda = \alpha \ , \ \kappa = \alpha+1)$,

b) $\omega = \alpha-2, \quad \kappa = \alpha-1, \quad \lambda = \alpha-2 \quad (\text{or } \lambda = \alpha-1, \ \kappa = \alpha-2)$,

c) $\omega = \alpha \ , \quad \kappa = \lambda$.

In view of the restrictions on the summation $\Sigma'_{k,\kappa;\ell,\lambda}$ we choose in the cases a) and b) one of both possibilities for κ and λ, e.g. the first ones; the summations over k and ℓ can then be performed without any restriction. Due to the fact that in case c) κ equals λ, k and ℓ have to be different there, and each couple (k,ℓ) may occur only once; we can satisfy these restrictions, e.g., by replacing $\Sigma'_{k,\kappa;\ell,\lambda}$ by $\Sigma^{(k<\ell)}_{k,\ell,\kappa}$, where the Latin indices are thought of as being ordered in some way. The total contribution from the cases a) and b) reduces to

$$2 \sum_{j,k,\ell} <\sigma_1\sigma_k><\sigma_k\sigma_\ell><\sigma_\ell\sigma_j> . \tag{31}$$

The individual contributions from case c) can be written as

$$<\sigma_{1\alpha}\sigma_{j\alpha}\sigma_{k\kappa}\sigma_{k\,\kappa+1}\sigma_{\ell\kappa}\sigma_{\ell\,\kappa+1}>_0 = \begin{cases} <\sigma_1\sigma_j\sigma_k\sigma_\ell><\sigma_k\sigma_\ell> & (\kappa = \alpha,\ \alpha-1), \\ <\sigma_1\sigma_j><\sigma_k\sigma_\ell>^2 & (\kappa \neq \alpha,\ \alpha-1). \end{cases} \tag{32}$$

For the second term in eq. (30) we have $\omega=\alpha$, $\kappa=\lambda$; the individual contributions are

$$<\sigma_{1\alpha}\sigma_{j\alpha}>_0<\sigma_{k\kappa}\sigma_{k\,\kappa+1}\sigma_{\ell\kappa}\sigma_{\ell\,\kappa+1}>_0 = <\sigma_1\sigma_j><\sigma_k\sigma_\ell>^2 \quad \text{for all } \kappa. \tag{33}$$

Substituting eqs. (32) and (33) into eq. (30) for a_2, and taking into account that in the two terms of eq. (30) the conditions on the summation for these contributions are the same, we find that the terms with $\kappa\neq\alpha$, $\alpha-1$ cancel pairwise, so that only the terms with $\kappa=\alpha$, $\alpha-1$ are left. This fact reflects the occurrence of only connected 2-graphs in the full graphical expansion of the susceptibility. It follows that the total contribution to a_2 from the terms considered in eqs. (32) and (33) is given by

$$2\sum_{j,k,\ell}^{(k<\ell)}\langle\sigma_1\sigma_j\sigma_k\sigma_\ell\rangle\langle\sigma_k\sigma_\ell\rangle - 2\sum_{j,k,\ell}^{(k<\ell)}\langle\sigma_1\sigma_j\rangle\langle\sigma_k\sigma_\ell\rangle^2 . \tag{34}$$

Combining the expressions (31) and (34) we find

$$a_2 = 2\sum_{j,k,\ell}\langle\sigma_1\sigma_k\rangle\langle\sigma_k\sigma_\ell\rangle\langle\sigma_\ell\sigma_j\rangle + 2\sum_{j,k,\ell}^{(k<\ell)}\langle\sigma_1\sigma_j\sigma_k\sigma_\ell\rangle\langle\sigma_k\sigma_\ell\rangle -$$

$$- 2\sum_{j,k,\ell}^{(k<\ell)}\langle\sigma_1\sigma_j\rangle\langle\sigma_k\sigma_\ell\rangle^2 . \tag{35}$$

Finally, we mention the result for a_3, which can be derived in a completely similar way as the expressions for a_0, a_1 and a_2:

$$a_3 = 2\sum_{j,k,\ell,m}\langle\sigma_1\sigma_k\rangle\langle\sigma_k\sigma_\ell\rangle\langle\sigma_\ell\sigma_m\rangle\langle\sigma_m\sigma_j\rangle + 2\sum_{j,k,\ell,m}^{(\ell<m)}\langle\sigma_1\sigma_k\sigma_\ell\sigma_m\rangle\langle\sigma_\ell\sigma_m\rangle\langle\sigma_k\sigma_j\rangle +$$

$$+ 2\sum_{j,k,\ell,m}^{(\ell<m)}\langle\sigma_1\sigma_k\rangle\langle\sigma_k\sigma_\ell\sigma_m\sigma_j\rangle\langle\sigma_\ell\sigma_m\rangle + 2\sum_{j,k,\ell,m}^{(k<\ell<m)}\langle\sigma_1\sigma_k\sigma_\ell\sigma_m\rangle\langle\sigma_k\sigma_\ell\sigma_m\sigma_j\rangle -$$

$$- 6\sum_{j,k,\ell,m}^{(\ell<m)}\langle\sigma_1\sigma_k\rangle\langle\sigma_k\sigma_j\rangle\langle\sigma_\ell\sigma_m\rangle^2 , \tag{36}$$

which can be simplified a little by noting that the second and the third sum, though different in origin, are equal.

In general, each coefficient a_n can be written as a sum of products of multiple-spin correlation functions on L_N'. Obviously, the structure of the a_n will become more complicated as n increases, due to the appearance of correlation functions involving more and more spins. However, one general property can be found immediately: for any n the points 1α, $k_1\kappa_1,\ldots,$ $k_n\kappa_n$, $j\,\omega$ can be chosen in such a way that they lie in the n+1 consecutive layers labelled α, $\alpha+1$, ..., $\alpha+n$ or α, $\alpha-1$, ..., $\alpha-n$. This yields for each n a contribution $2\Sigma_{j,k_1,\ldots,k_n}\langle\sigma_1\sigma_{k_1}\rangle\langle\sigma_{k_1}\sigma_{k_2}\rangle\ldots\langle\sigma_{k_n}\sigma_j\rangle$ to a_n. The significance of this fact, together with some properties of the coefficients a_0, a_1, a_2 and a_3, will be discussed in the next section.

On account of the periodic boundary condition with respect to the d^{th} lattice axis the expressions for the a_n in terms of multiple-spin correlation functions on $L'_{N'}$ are independent of N_d provided N_d is larger than n. In the derivation of the expressions for a_n, $n\leq 3$, we have tacitly assumed that this condition is satisfied. In order to obtain the thermodynamic limit of χ'_d we have to let the numbers $N_1,\ldots,N_d$ tend to infinity. The limit $N_d \to \infty$ is automatically taken if we substitute for each a_n the expression in terms of multiple-spin correlation functions on $L'_{N'}$ valid for all $N_d > n$. Taking the other limits $N_1,\ldots,N_{d-1} \to \infty$ implies that for each of these correlation functions the thermodynamic limit in $L'_{N'}$ is taken. Henceforth, we shall denote the power series for the thermodynamic limit of χ'_d also by $\Sigma_n a_n t_d^n$.

§ 3. Some properties of the coefficients a_n

As remarked in § 2, the coefficients a_n in the series $\Sigma_n a_n t_d^n$ for χ'_d have the property in common that they are built up entirely from multiple-spin correlation functions on the (d-1)-dimensional lattice L'. Since the critical temperature of the Ising system on the d-dimensional lattice L approaches, in the limit of extreme anisotropy under consideration, the critical temperature of the Ising system on L', the investigation of the critical behaviour of χ'_d leads automatically to a study of the critical properties of the a_n. With a view to the fact that we are mainly interested in the critical equation of L, which by the symmetry properties of Ising systems on Cartesian lattices mentioned in IV, § 2 can, for arbitrary signs of the t_i, be derived immediately from the equation for the case where all t_i are positive, we restrict ourselves to the ferromagnetic case. An advantage of this restriction is that it permits us the use of Griffiths' inequalities for multiple-spin correlation functions [5], which are generally valid only for ferromagnetic systems; they will turn out to form an important tool for the investigation of the a_n.

As noted in § 2, the coefficient a_0 is just the susceptibility χ'_{d-1}, which in combination with eq. (1) implies for its critical behaviour

$$a_0(=\chi'_{d-1}) \sim \Delta_{d-1}^{-\gamma_{d-1}} \qquad \left[(t_1,\ldots,t_{d-1}) \to (t_{1c},\ldots,t_{d-1\,c})\right]. \tag{37}$$

In other words, the critical exponent $\gamma^{(0)}$ of a_0 is equal to γ_{d-1}:

$$\gamma^{(0)} = \gamma_{d-1}. \tag{38}$$

Expression (29) for a_1 can be rewritten by first performing the summation over j, which yields for each value of k a factor χ'_{d-1}, and hence transforms eq. (29) into

$$a_1 = 2\,\left(\sum_k \langle\sigma_1\sigma_k\rangle\right)\chi'_{d-1}. \tag{39}$$

Performing in this equation the summation over k we obtain

$$a_1 = 2\,\chi'^{\,2}_{d-1}\;. \tag{40}$$

Hence the critical behaviour of a_1 can, just as that of a_0, be described in terms of the behaviour of χ'_{d-1}, viz. as

$$a_1 \sim \Delta_{d-1}^{-2\gamma_{d-1}} \qquad \left[(t_1,\ldots,t_{d-1}) \to (t_{1c},\ldots,t_{d-1\,c})\right]. \tag{41}$$

Consequently, the critical exponent $\gamma^{(1)}$ of a_1 is twice γ_{d-1}:

$$\gamma^{(1)} = 2\gamma_{d-1}. \tag{42}$$

The situation is less simple for the a_n with $n\geq 2$. By the occurrence of correlation functions involving more and more spins these coefficients cannot be expressed in terms of χ'_{d-1} alone, as was the case for a_0 and a_1. However, using Griffiths' inequalities we can still obtain information on the critical behaviour of the a_n for at least n=2,3.

For the investigation of a_2 we start by rewriting eq. (35) in an obvious short-hand notation as

$$a_2 = 2\sum_{j,k,\ell} \langle 1k\rangle\langle k\ell\rangle\langle \ell j\rangle + 2\sum_{j,k,\ell}^{(k<\ell)} \langle 1jk\ell\rangle\langle k\ell\rangle - 2\sum_{j,k,\ell}^{(k<\ell)} \langle 1j\rangle\langle k\ell\rangle^2 \equiv$$

$$\equiv 2a_{2a} + 2a_{2b} - 2a_{2c}. \tag{43}$$

By an argument similar to that which led to eq. (40), the quantity a_{2a} is seen to be equal to $\chi'^{\,3}_{d-1}$, so that it behaves as $\Delta_{d-1}^{-3\gamma_{d-1}}$ for $(t_1,\ldots,t_{d-1}) \to$ $\to (t_{1c},\ldots,t_{d-1\,c})$. To prove that a_2 diverges also as $\Delta_{d-1}^{-3\gamma_{d-1}}$ in this limit, it is sufficient to show that $a_{2b}-a_{2c}$ is non-negative and not larger than a_{2a}, so that it does not diverge stronger than a_{2a}. By Griffiths' generalized first inequality applied to the two-spin correlation function $\langle k\ell\rangle$,

$$\langle k\ell\rangle \geq 0, \tag{44}$$

in combination with Griffiths' generalized second inequality on the four-spin correlation function $\langle 1jk\ell\rangle$,

$$\langle 1jk\ell\rangle - \langle 1j\rangle\langle k\ell\rangle \geq 0, \tag{45}$$

no term in $a_{2b}-a_{2c}$ is negative; hence $a_{2b}-a_{2c}$ itself is non-negative. It follows that a_2 is not less than $2a_{2a}$, which, as mentioned above, is equal to the (non-negative) quantity $2\chi'^{\,3}_{d-1}$; i.e.,

$$a_2 \geq 2\chi'^{\,3}_{d-1} \geq 0, \tag{46}$$

which implies

$$\gamma^{(2)} \geq 3\gamma_{d-1}. \tag{47}$$

In order to prove that $a_{2b}-a_{2c}$ does not diverge stronger than a_{2a} we write $(a_{2b}-a_{2c}) - a_{2a}$ as

$$(a_{2b}-a_{2c}) - a_{2a} = \sum_{j,k,\ell}^{(k<\ell)} \Big[\langle 1jk\ell\rangle - \langle 1j\rangle\langle k\ell\rangle - \langle 1k\rangle\langle \ell j\rangle - \langle 1\ell\rangle\langle kj\rangle \Big] \langle k\ell\rangle -$$

$$- \sum_{j,k} \langle 1k\rangle\langle kk\rangle\langle kj\rangle. \tag{48}$$

By the inequality

$$\langle 1jk\ell\rangle - \langle 1j\rangle\langle k\ell\rangle - \langle 1k\rangle\langle \ell j\rangle - \langle 1\ell\rangle\langle kj\rangle \leq 0, \tag{49}$$

which is a weaker form of Griffiths' third inequality, no term in the first sum of eq. (48) is positive, so that the sum itself is not positive either; obviously, the second sum in eq. (48), being equal to $\chi'^{\,2}_{d-1}$, is non-negative. It follows that $a_{2b}-a_{2c}$ is not larger than a_{2a}. Hence

$$a_2 \leq 4\chi'^{\,3}_{d-1}, \tag{50}$$

and consequently, by the non-negativity of a_2,

$$\gamma^{(2)} \leq 3\gamma_{d-1}. \tag{51}$$

Combining eqs. (46) and (50) we conclude that the critical behaviour of a_2 is the same as that of $\chi'^{\,3}_{d-1}$:

$$a_2 \sim \Delta_{d-1}^{-3\gamma_{d-1}} \qquad \Big[(t_1,\ldots,t_{d-1}) \rightarrow (t_{1c},\ldots,t_{d-1\,c}) \Big], \tag{52}$$

in other words,

$$\gamma^{(2)} = 3\gamma_{d-1}. \tag{53}$$

For the investigation of a_3 we rewrite eq. (36) in the same notation as used in eq. (43):

$$a_3 = 2\sum_{j,k,\ell,m} \langle 1k\rangle\langle k\ell\rangle\langle \ell m\rangle\langle mj\rangle + 4\sum_{j,k,\ell,m}^{(\ell<m)} \langle 1k\ell m\rangle\langle \ell m\rangle\langle kj\rangle +$$

$$+ 2\sum_{j,k,\ell,m}^{(k<\ell<m)} \langle 1k\ell m\rangle\langle k\ell mj\rangle - 6\sum_{j,k,\ell,m}^{(\ell<m)} \langle 1k\rangle\langle kj\rangle\langle \ell m\rangle^2 \equiv 2a_{3a} + 4a_{3b} + 2a_{3c} - 6a_{3d}, \tag{54}$$

where we have taken into account that the second and the third sum in eq. (36) are equal. The first term in this expression is equal to $2\chi'^{4}_{d-1}$. In contrast with the situation for a_2 we cannot show that the remainder in a_3, viz. the quantity $4a_{3b}+2a_{3c}-6a_{3d}$, is non-negative, which would for the critical exponent $\gamma^{(3)}$ of a_3 yield immediately the inequality $\gamma^{(3)} \geq 4\gamma_{d-1}$. On the contrary, for T=0, where all multiple-spin correlation-functions are unity, it is negative, since the products of multiple-spin correlation functions occurring with a -sign outnumber those occurring with a +sign. We can prove, however, that a_3 is a non-negative quantity which does not diverge faster than χ'^{4}_{d-1} if the Curie temperature is approached. To this end we first consider the quantity $a_{3a}+a_{3c}-a_{3d}$, which we write as

$$
\begin{aligned}
a_{3a}+a_{3c}-a_{3d} = \sum_{j,k,\ell,m}^{(k<\ell<m)} \Big\{ & \langle 1k\ell m\rangle\langle k\ell mj\rangle + \\
& + \langle 1k\rangle\langle \ell m\rangle\Big[\langle k\ell\rangle\langle mj\rangle+\langle km\rangle\langle \ell j\rangle-\langle kj\rangle\langle \ell m\rangle\Big] + \langle 1\ell\rangle\langle km\rangle\Big[\langle k\ell\rangle\langle mj\rangle+\langle \ell m\rangle\langle kj\rangle-\langle \ell j\rangle\langle km\rangle\Big] + \\
& + \langle 1m\rangle\langle k\ell\rangle\Big[\langle km\rangle\langle \ell j\rangle+\langle \ell m\rangle\langle kj\rangle-\langle mj\rangle\langle k\ell\rangle\Big]\Big\} + \sum_{j,k,m}^{(k\neq m)} \langle 1k\rangle\langle kk\rangle\langle km\rangle\langle mj\rangle + \\
& + \sum_{j,k,\ell}^{(k\neq \ell)} \langle 1k\rangle\langle k\ell\rangle\langle \ell k\rangle\langle kj\rangle + \sum_{j,k,\ell}^{(k\neq \ell)} \langle 1k\rangle\langle k\ell\rangle\langle \ell\ell\rangle\langle \ell j\rangle + \\
& + \sum_{j,k} \langle 1k\rangle\langle kk\rangle\langle kk\rangle\langle kj\rangle - \sum_{j,k,m}^{(k<m)} \langle 1k\rangle\langle kj\rangle\langle km\rangle^2 - \sum_{j,k,\ell}^{(\ell<k)} \langle 1k\rangle\langle kj\rangle\langle k\ell\rangle^2 . \qquad (55)
\end{aligned}
$$

In the analysis of the first sum in this expression we consider for each set of values of j, k, ℓ and m one of the quantities $\langle kj\rangle\langle \ell m\rangle$, $\langle \ell j\rangle\langle km\rangle$ and $\langle mj\rangle\langle k\ell\rangle$ which is larger than (or equal to) the other two. For instance, let this quantity be $\langle kj\rangle\langle \ell m\rangle$, i.e. the differences $\langle \ell m\rangle\langle kj\rangle-\langle \ell j\rangle\langle km\rangle$ and $\langle \ell m\rangle\langle kj\rangle-\langle mj\rangle\langle k\ell\rangle$ are non-negative. Using furthermore that by Griffiths' generalized second inequality the difference $\langle 1k\ell m\rangle\langle k\ell mj\rangle-\langle 1k\rangle\langle \ell m\rangle\langle kj\rangle\langle \ell m\rangle$ is non-negative, we find that each term in the first sum of eq. (55), and hence the sum itself, is non-negative. Replacing in the second last sum in eq. (55) the dummy index m by ℓ we see that the last two sums yield together just the third sum in eq. (55), so that these sums cancel. Obviously, the remaining terms are non-negative. Summarizing, we conclude that $a_{3a}+a_{3c}-a_{3d}$ is non-negative:

$$a_{3a} + a_{3c} - a_{3d} \geq 0 . \qquad (56)$$

The same holds, again by Griffiths' generalized second inequality, for $a_{3b}-a_{3d}$:

$$a_{3b} - a_{3d} \geq 0 \; . \tag{57}$$

Combining eqs. (56) and (57) we find that a_3 is non-negative.

In order to prove that a_3 does not diverge faster than χ'^{4}_{d-1}, we first notice that a_{3c} is not larger than a_{3b}:

$$a_{3c} \leq a_{3b} \; . \tag{58}$$

This can be proven by writing $a_{3c}-a_{3b}$ as

$$a_{3c}-a_{3b} = \sum_{j,k,\ell,m}^{(k<\ell<m)} <1k\ell m> \Big[<k\ell mj> - <\ell m><kj> - <km><\ell j> - <k\ell><mj> \Big] -$$

$$- \sum_{j,k,m}^{(k<m)} <1kkm><km><kj> - \sum_{j,k,\ell}^{(\ell<k)} <1k\ell k><\ell k><kj>, \tag{59}$$

which by Griffiths' inequalities is not larger than zero. Next we write $a_{3b}-a_{3d}$ as

$$\chi'_{d-1} \sum_{k,\ell,m}^{(\ell<m)} \Big[<1k\ell m> - <1k><\ell m> \Big] <\ell m> .$$

The sum occurring in this expression is identical to the quantity $a_{2b}-a_{2c}$, which we proved to be non-negative and not larger than χ'^{3}_{d-1}. Consequently,

$$0 \leq a_{3b} - a_{3d} \leq \chi'^{4}_{d-1} \; , \tag{60}$$

which implies, in combination with eq. (58)

$$a_{3c} - a_{3d} \leq \chi'^{4}_{d-1} \; . \tag{61}$$

Combining eqs. (60) and (61) with the fact that a_{3a} is equal to χ'^{4}_{d-1}, we find for a_3 (which was shown above to be non-negative):

$$0 \leq a_3 = 2a_{3a} + 4(a_{3b}-a_{3d}) + 2(a_{3c}-a_{3d}) \leq 8\chi'^{4}_{d-1} . \tag{62}$$

Consequently, a_3 does not diverge faster than $\Delta_{d-1}^{-4\gamma_{d-1}}$, and hence

$$\gamma^{(3)} \leq 4\gamma_{d-1} \; . \tag{63}$$

In view of the increasingly complicated structure of the a_n we have not attempted to obtain information on their critical behaviour for larger values of n. The presence of a term $2\Sigma_{j,k_1,\ldots,k_n} <1k_1><k_1k_2>\ldots<k_nj> = 2\chi'^{n+1}_{d-1}$ in a_n leads us to conjecture that for all n

$$a_n \sim \Delta_{d-1}^{-(n+1)\gamma_{d-1}} \left[(t_1,\ldots,t_{d-1}) \to (t_{1c},\ldots,t_{d-1\,c})\right], \tag{64}$$

and hence

$$\gamma^{(n)} = (n+1)\gamma_{d-1}, \tag{65}$$

which would be in line with the results obtained thus far for $n\leq 3$. The various arguments in favour of this conjecture which may be drawn from scaling theory and numerical results have been mentioned in the introduction. An additional support comes from the results of I, where we considered the crossover from d=2 (the quadratic lattice) to d=1 (the linear chain). Since the critical behaviour of the susceptibility χ_1' of the linear chain is given by

$$\chi_1' \sim (1-t_1)^{-1} \qquad (t_1\uparrow 1), \tag{66}$$

the critical exponent γ_1 equals 1; on the other hand, the coefficients $a_n(t_1)$ in the series $\Sigma_n a_n(t_1)t_2^n$ for χ_2' are proportional to $(1-t_1)^{-(n+1)}$ for $t_1\uparrow 1$ *) (see I, eq. (38)). Hence for the case d=2 eqs. (64) and (65) are established for all n.

In the next section, devoted to a study of the critical equation of the simple cubic lattice, we shall assume that eq. (64) does indeed hold for all n.

§ 4. The critical equation of the extremely anisotropic simple cubic lattice

The results of § 3 enable us to pursue the investigation of the critical equation of the simple cubic (s.c.) lattice, which we started numerically in II. In conformity with eq. (19) we write for the susceptibility

$$\chi_3' = \sum_{n=0}^{\infty} a_n(t_1,t_2)t_3^n, \tag{67}$$

where we have explicitly indicated the dependence of the coefficients a_n on the variables t_1 and t_2. We assume that the $a_n(t_1,t_2)$ are positive quantities, which has for $n\leq 3$ been established in the foregoing section. Then the equation which expresses the radius of convergence of the series (67) in terms of t_1 and t_2 amounts just to the critical equation

$$\Delta_3(t_1,t_2,t_3) = 0. \tag{68}$$

In other words, the critical equation is equivalent to the equation

$$t_3 = \lim_{n\to\infty} \frac{a_{n-1}(t_1,t_2)}{a_n(t_1,t_2)}. \tag{69}$$

Equation (68) describes the critical surface in the t_1,t_2,t_3-space. For

*) provided the numbers b_{n0} differ from zero, which is strongly suggested by the numerical evidence.

$t_3=0$ it reduces to the equation (16) for the critical line in the t_1,t_2-plane of the quadratic lattice. To see how this reduction comes about if t_3 approaches zero (in other words, to study the form of the critical surface near the t_1,t_2-plane) we have to consider the critical behaviour of the coefficients $a_n(t_1,t_2)$. According to the conjecture made in the previous section, this critical behaviour is for all n the same as that of the $n+1^{st}$ power of the susceptibility χ_2' of the quadratic lattice; taking into account that the critical exponent γ_2 of χ_2' (see eq. (1)) is equal to 7/4 we may therefore write

$$a_n(t_1,t_2) = b_n(t_1,t_2)\, \Delta_2^{-(n+1)7/4}(t_1,t_2), \tag{70}$$

where $b_n(t_1,t_2)$ tends to a finite ($\neq 0, \pm\infty$) value if a point (t_{1c},t_{2c}) of the critical line (16) is approached. Substituting eq. (70) into eq. (69) we obtain the critical equation of the s.c. lattice in the form

$$t_3 = \left[\lim_{n\to\infty} \frac{b_{n-1}(t_1,t_2)}{b_n(t_1,t_2)}\right] \Delta_2^{7/4}(t_1,t_2), \tag{71}$$

which we write as

$$t_3 = \left[b(t_1,t_2)\right]^{-1} \Delta_2^{7/4}(t_1,t_2). \tag{72}$$

If we assume that with the $b_n(t_1,t_2)$ also the function $b(t_1,t_2)$ tends to a finite value if the critical line (16) is approached, this equation implies that for small values of Δ_2, t_3 becomes proportional to a power of Δ_2 which exceeds unity (viz. 7/4); equivalently, the critical surface touches the t_1,t_2-plane, and hence, by symmetry also the t_1,t_3- and the t_2,t_3-plane.

In order to lay the connection between the results of this paper and those of II, we finally investigate the asymptotic form of the critical equation if not only t_3 but also t_2 tends to zero, i.e. the form of the critical surface near the point (1,0,0).

If the statement that the critical surface touches the coordinate planes is correct, this implies that in the point (1,0,0) - and similarly in the points (0,1,0) and (0,0,1) - it has at least two tangent planes, viz. the two coordinate planes in which this point lies. This property of the critical surface is consistent with the conical behaviour to which we concluded in II, § 3 on the basis of numerical results. We now substitute eq. (16) into eq. (72) and rewrite the equation thus obtained as

$$\left(1-(1+t_1)\frac{t_2}{1-t_1}\right)^{7/4}-\frac{b(t_1,t_2)}{(1-t_1)^{3/4}}\frac{t_3}{1-t_1}=0. \tag{73}$$

According to part a) of the leading-order hypothesis (IV, § 3) the asymptotic form of the critical equation in the limit $t_2,t_3 \to 0$ should be the "critical equation" of the series

$$\sum_{m,n=0}^{\infty} b_{mn0}\, x_2^m x_3^n \qquad \left(x_{2(3)}=\frac{t_{2(3)}}{1-t_1}\right) \tag{74}$$

occurring in the leading-order term of χ_3' for this limit, i.e. the equation which is satisfied by those combinations of values of x_2 and x_3 for which this series becomes singular. Consequently, the critical equation (73) becomes in the limit of extreme anisotropy considered here an equation of the form

$$(1-2x_2)^{7/4}-\alpha(x_2)x_3=0; \tag{75}$$

in other words, $b(t_1,t_2)$ behaves for $t_2 \ll 1$, $1-t_1 \ll 1$ as the product of a function $\alpha(x_2)$ depending on x_2 alone, and the function $(1-t_1)^{3/4}$.

On the analogy of the procedure followed in the investigation of the asymptotic form of eq. (63) for the case $t_3 \approx 0$, we may investigate the asymptotic form of eq. (75) for the case $x_3 \approx 0$ by considering the analytic properties of the "coefficients" $B_n(x_2) \equiv \Sigma_m b_{mn0}\, x_2^m$ of x_3^n occurring in the series (74) viewed as a series in x_3. The coefficient $B_0(x_2)$ is just the quantity $\Sigma_n b_{n0}\, x_2^n$ studied in I and IV in the investigation of the susceptibility of the quadratic lattice with $(t_1,t_2) \approx (1,0)$. According to part a) of the leading-order hypothesis, $B_0(x_2)$ (which in IV, §3 was denoted by $B(x_2)$) becomes singular if x_2 approaches the "critical value" $\frac{1}{2}$ corresponding to the asymptotic form (cf. I, eq. (9))

$$1-2x_2=0 \tag{76}$$

of the critical equation (16) of the quadratic lattice; by part c) of the leading-order hypothesis, its singular behaviour for $x_2 \uparrow \frac{1}{2}$ is of the same type as the critical behaviour of the susceptibility χ_2' of the quadratic lattice:

$$B_0(x_2) \sim (1-2x_2)^{-7/4} \qquad (x_2 \uparrow \tfrac{1}{2}). \tag{77}$$

The coefficient $B_1(x_2)$ is related to $B_0(x_2)$ as

$$B_1(x_2)=2\left[B_0(x_2)\right]^2 ; \tag{78}$$

this relation is analogous to, and can be derived in the same way as, the relation $a_1=2\chi'^{\,2}_{d-1}=2a_0^2$, cf. eqs. (37) and (40). Hence the critical behaviour of $B_1(x_2)$ is given by

$$B_1(x_2) \sim (1-2x_2)^{-2\cdot 7/4} \qquad (x_2\uparrow\tfrac{1}{2}). \tag{79}$$

Eqs. (77) and (79) are completely analogous to eqs. (37) and (41) (with d=2) for a_0 and a_1, respectively, with the replacement of the variable $\Delta_2(t_1,t_2)$ by the variable $1-2x_2$. On the analogy of the conjecture made on the generalization of the latter equations to general n, eq. (64), we make the conjecture that the critical behaviour of the functions $B_n(x_2)$ for general n is given by

$$B_n(x_2) \sim (1-2x_2)^{-(n+1)7/4} \qquad (x_2\uparrow\tfrac{1}{2}). \tag{80}$$

Starting from this equation and following a procedure similar to that leading to eq. (72), we arrive at the statement that the asymptotic form of eq. (75) for $x_2\uparrow\frac{1}{2}$ is

$$x_3 \sim (1-2x_2)^{7/4} \qquad (x_2\uparrow\tfrac{1}{2}), \tag{81}$$

which implies that $\alpha(x_2)$ should be finite in this limit. Let us now, starting from this property of eq. (75), look for a possible explicit form for this equation. The requirements which in addition to eq. (81) we impose on this form are

1) that it be symmetric in the variables x_2 and x_3;
2) that it be consistent with the numerical results obtained in II, § 3 for the cases $t_2=at_3$, a=1,2,4,6 and 8.

The simplest equation consistent with eq. (81) and the first requirement is

$$(1-2x_2)^{7/4} + (1-2x_3)^{7/4} = 1; \tag{82}$$

this equation would imply that $\lim_{x_2\to\frac{1}{2}} \alpha(x_2) = \frac{7}{2}$. Putting in eq. (82) x_3 equal to x_2/a yields the equation

$$(1-2x_2)^{7/4} + (1-2x_2/a)^{7/4} = 1, \tag{83}$$

the solution of which we denote by $x_2(a)$. If eq. (82) is the correct asymptotic critical equation, $x_2(a)$ is, by the definition of this equation, the radius of convergence $b^{-1}(a)$ of the series $\Sigma_{n=0}^{\infty} b_{n0}(a)x_2^n$ obtained from the series (74) by substituting $x_3=x_2/a$ (cf. II, eq. (6)). In order to check if the second requirement is fulfilled, we have therefore to compare the numbers $x_2^{-1}(a)$ with the respective numerical estimates for the quantity $b(a)$. These two sets of numbers are listed in table I, together with the values of $2(1+\frac{1}{a})$, to which b(a) would be equal if the critical surface had a unique tangent plane in (1,0,0) (cf. II, eq. (12)).

TABLE I

a	b(a)	$x_2^{-1}(a)$	$2(1+\frac{1}{a})$
1	6.10 ± 0.02	6.115...	4
2	4.48^5 ± 0.02	4.491...	3
4	3.54 ± 0.02	3.544...	$2\frac{1}{2}$
6	3.19 ± 0.02	3.179...	$2\frac{1}{3}$
8	2.99 ± 0.02	2.977...	$2\frac{1}{4}$

The agreement between b(a) and $x_2^{-1}(a)$ is very close, and in sharp contrast to the disagreement between b(a) and $2(1-\frac{1}{a})$. Eq. (82) may therefore be considered to be a good candidate for the asymptotic form of the critical equation near (1,0,0), at least as far as our information obtained in II and in the present paper is concerned.

One may try to find other equations which also imply eq. (81) and satisfy the requirements 1) ans 2); from the agreement between the values for b(a) and $x_2^{-1}(a)$ it is to be expected that a possible correction on eq. (82) will be relatively small. As a specific example, we have considered an equation which differs from eq. (82) in that it also contains terms with an additional factor $1-2x_2$ or $1-2x_3$, respectively:

$$(1-2x_2)^{7/4} + (1-2x_3)^{7/4} + \lambda(1-2x_2)^{11/4} + \lambda(1-2x_3)^{11/4} = 1 + \lambda. \qquad (84)$$

Adapting the value of λ to the various estimates for b(a) we find not only that the values thus obtained are very small, with an uncertainty exceeding their absolute value (e.g. $\lambda = -0.006 \pm 0.008$ for a=1, $\lambda = -0.004 \pm 0.018$ for a=4), but also that their scattering over the "range" a=1,2,4,6,8 is of the same order of magnitude. In view of these facts we will, as long as no further data invalidate eq. (82) in an unambiguous way, stick to the conjecture that this equation is the asymptotic form of the critical equation for $(t_1,t_2,t_3) \rightarrow$ $\rightarrow (1,0,0)$. In order to compare the foregoing results with those following from an approximate theory such as the Bethe-Peierls approximation , we note that the asymptotic form, in the limit considered, of the critical equation of the s.c. lattice in this approximation (cf. II, eqs. (13) and (14)):

$$1 - 4x_2 - 4x_3 = 0, \tag{85}$$

can also be obtained after the application of a procedure similar to the one leading to eq. (82); to that end we have to take for the susceptibility exponent γ_2 the Bethe-Peierls value 1, and for the asymptotic form of the critical equation of the quadratic lattice (cf. I, eq. (14)) the equation

$$1 - 4x_2 = 0. \tag{86}$$

Concerning the direct relevance of the foregoing results to the Curie temperature of the s.c. lattice, $T_c(J_1,J_2,J_3)$ itself, we notice that by eq. (72) $T_c(J_1,J_2,J_3)$ reduces, for J_3 tending to zero, to the Curie temperature of the quadratic lattice with a difference vanishing like $J_3^{4/7}$. The generalization of eq. (72) to arbitrary values of d, with the power 7/4 replaced by γ_{d-1}, implies for the difference $T_c(J_1,\ldots,J_d) - T_c(J_1,\ldots,J_{d-1},0)$ a proportionality to $J_d^{1/\gamma_{d-1}}$. This feature was predicted by Abe [11] on the basis of scaling arguments.

In conclusion we want to stress that the results obtained in this paper for the form of the critical surface near the coordinate planes, and in particular near the coordinate axes, show that the full critical equation for the s.c. lattice must be of a fairly complicated nature, in contrast with the critical equation for the linear chain, I, eq. (8), and for the quadratic lattice, eq. (16). In this respect the exact critical equation of the s.c. lattice differs drastically from that found in the Bethe-Peierls approximation (cf. II, eq. (13)):

$$1 - (t_1+t_2+t_3) - 3(t_1t_2 + t_1t_3 + t_2t_3) - 5t_1t_2t_3 = 0\ , \tag{87}$$

which is of the same nature as the corresponding equations for the linear chain (which is again I, eq. (8)) and for the quadratic lattice, I, eq. (13).

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SAMENVATTING

In dit proefschrift worden een aantal thermodynamische eigenschappen bestudeerd van spin-$\frac{1}{2}$ Ising-systemen op extreem anisotrope Cartesische (=hyperkubische) roosters, d.w.z. Cartesische roosters waarin de koppelingsconstanten J_i (die een maat zijn voor de sterkte van de wisselwerking tussen naaste buren) voor de naaste buren langs één of meer roosterassen veel kleiner zijn dan die voor de naaste buren langs de andere roosterassen. Doel van dit onderzoek is een inzicht te verkrijgen in de manier waarop thermodynamische grootheden van Ising-systemen op een gegeven rooster L overgaan in die voor een rooster van lagere dimensie L', wanneer het oorspronkelijke rooster door het nul worden van de "kleine" koppelingsconstanten uiteenvalt in een aantal, onderling ongekoppelde, roosters van deze lagere dimensie.

Het onderzoek wordt uitgevoerd aan de hand van reeksontwikkelingen bij hoge temperatuur voor de initiële susceptibiliteit. Met het oog op de extreme anisotropie wordt de standaardprocedure van reeksontwikkeling naar alle variabelen $t_i = \tanh J_i/kT$ vervangen door een reeksontwikkeling naar slechts die t_i die corresponderen met de kleine koppelingsconstanten. De coëfficiënten in zo'n reeks kunnen algemeen worden uitgedrukt in termen van spincorrelatiefuncties op het bijbehorende stelsel ongekoppelde (lager-dimensionale) roosters L'. Op grond van dit feit zal men verwachten dat de coëfficiënten een singulier gedrag vertonen bij de kritische temperatuur van L'. Deze temperatuur is echter niet de kritische temperatuur van L, waaruit volgt dat de kritische eigenschappen van de coëfficiënten niet representatief zullen zijn voor die van de beschouwde thermodynamische grootheid, i.c. de susceptibiliteit. Anderzijds zullen, naarmate de kritische temperatuur van L en die van L' dichter bij elkaar komen te liggen (hetgeen bij toenemende anisotropie het geval is), de eerstgenoemde kritische eigenschappen een steeds grotere rol gaan spelen. Op grond van deze overweging hebben wij in de beschouwde reeksen elke coëfficiënt vervangen door dat deel dat de dominante singulariteit vertoont. De zo ontstane grootheid noemen wij de leidende-orde term (van de susceptibiliteit) voor het beschouwde geval van extreme anisotropie. Aan de bestudering van deze grootheid is het grootste deel van dit proefschrift gewijd.

In hoofdstuk I wordt het ferromagnetische kwadratische rooster beschouwd. Van de leidende-orde term van de susceptibiliteit worden de eerste vijf coëfficiënten berekend. Uit de algemene structuur van de leidende-orde term blijkt dat het kritische gedrag verschillende facetten heeft: al naar gelang de omstandigheden vertoont de leidende-orde term het kritische gedrag van de susceptibiliteit van de lineaire keten, of een gedrag dat, binnen de foutenmarge, hetzelfde is als dat van de susceptibiliteit van het kwadratische rooster, of zelfs een willekeurig intermediair gedrag. Bovendien blijkt dat men uit de leidende-orde term, althans binnen de foutenmarge, de asymptotische vorm van de bekende uitdrukking voor de kritische temperatuur, gezien als een vergelijking in t_1 en t_2 (de "kritische vergelijking") kan vinden.

In hoofdstuk II wordt het ferromagnetische enkelvoudig kubische rooster beschouwd, waarin de koppelingsconstanten langs twee roosterassen veel kleiner zijn dan die langs de derde roosteras. De gevolgde procedure is een rechtstreekse generalisatie van die van hoofdstuk I. De conclusies over het kritische gedrag van de leidende-orde term van de susceptibiliteit zijn analoog met de aldaar gevonden resultaten. Stellen wij, naar analogie van de bevindingen van hoofdstuk I, dat de asymptotische kritische vergelijking van het enkelvoudig kubische rooster kan worden gevonden uit de leidende-orde term, dan volgt uit een (numerieke) analyse dat deze vergelijking zeker niet een multilineaire vergelijking in de variabelen t_i kan zijn, dit in tegenstelling tot de situatie voor het kwadratische rooster. In samenhang hiermee gedraagt het "kritische oppervlak", dat de kritische vergelijking in de t_1,t_2,t_3-ruimte beschrijft, zich in de nabijheid van de punten (1,0,0), (0,1,0) en (0,0,1) eerder als een kegel dan als een plat vlak.

In hoofdstuk III wordt de methode, gevolgd in de hoofdstukken I en II, gewijzigd met behulp van de generalisatie tot anisotrope roosters van een formele uitdrukking voor de susceptibiliteit van isotrope roosters, die oorspronkelijk door Sykes is geponeerd en later is afgeleid door Nagle en Temperley. De nieuwe methode is veel minder tijdrovend en veel overzichtelijker dan die van de hoofdstukken I en II.

In hoofdstuk IV worden antiferromagnetische Ising-systemen op Cartesische roosters beschouwd waarin de koppelingsconstante langs één roosteras (in absolute waarde) veel groter is dan die langs de andere roosterassen. Voor het geval dat de grote koppelingsconstante negatief is wordt met behulp van de in hoofdstuk III gevonden uitdrukking bewezen dat de leidende-orde term van de susceptibiliteit de helft is van de leidende-orde term van de grootheid $1+\langle\sigma_1\sigma_2\rangle$, waarbij $\langle\sigma_1\sigma_2\rangle$ de spin-spin correlatiefunctie is voor twee naaste buren langs de roosteras met de grote koppelingsconstante. Via deze relatie wordt voor het kwadratische rooster in dit geval van extreme anisotropie een gesloten uitdrukking afgeleid voor de leidende-orde term van de susceptibiliteit. De conclusies over kritisch gedrag zijn analoog aan die van de hoofdstukken I en II en geven extra steun aan het door andere onderzoekers geuite vermoeden dat susceptibiliteit en inwendige energie van antiferromagnetische Ising-systemen soortgelijk kritisch gedrag bezitten.

Hoofdstuk V is gewijd aan een onderzoek van de Curie-temperatuur van het enkelvoudig kubische rooster waarin de koppelingsconstante langs één of twee roosterassen veel kleiner is dan de andere koppelingsconstante(n). De bevindingen, verkregen op grond van analytische resultaten en min of meer voor de hand liggende vermoedens, zijn in overeenstemming met conclusies die volgen uit de schaaltheorie van kritisch gedrag.

Het in hoofdstuk II genoemde kritische oppervlak raakt volgens de verkregen resultaten aan de coördinaatvlakken, hetgeen consistent is met het kegelachtige gedrag nabij de punten (1,0,0), (0,1,0) en (0,0,1) waartoe in hoofdstuk II geconcludeerd was op grond van numerieke gegevens. Uitgaande hiervan wordt een expliciete vergelijking voorgesteld voor de asymptotische kritische vergelijking nabij het punt (1,0,0), die in uitstekende overeenstemming is met de numerieke resultaten van hoofdstuk II.

Op verzoek van de faculteit der wiskunde en natuurwetenschappen volgt hier een kort overzicht van mijn studie.

Na het behalen van het einddiploma Gymnasium β aan het Gymnasium Haganum te 's-Gravenhage begon ik in 1958 mijn studie in de wis- en natuurkunde aan de Rijksuniversiteit te Leiden. In oktober 1961 legde ik het candidaatsexamen d' (natuurkunde, wiskunde, scheikunde) af en in juli 1964 het doctoraalexamen theoretische natuurkunde met als bijvakken wiskunde en klassieke mechanica. Voor het laatstgenoemde examen legde ik tentamens af over colleges gegeven door Prof.Dr. J.A.M. Cox, Prof.Dr. S.R. de Groot en Prof.Dr. P. Mazur voor wat betreft de natuurkunde, en bij Prof.Dr. H.D. Kloosterman en Prof.Dr. C. Visser voor wat betreft de wiskunde.

Van februari 1964 tot medio februari 1973 ben ik verbonden geweest aan het Instituut-Lorentz, achtereenvolgens als candidaatassistent, doctoraalassistent en wetenschappelijk medewerker. Van mijn doctoraalexamen af verrichtte ik onderzoek onder leiding van Prof.Dr. P.W. Kasteleyn. Aanvankelijk had dit onderzoek betrekking op vibratiespectra van en elektronentoestanden en stochastische wandelingen in ongeordende kristallen, met name lineaire ketens; in september 1967 werd een aanvang gemaakt met het onderzoek waarvan de resultaten in dit proefschrift zijn beschreven. In de periode van juni 1964 af heb ik deelgenomen aan verschillende zomerscholen en conferenties op het gebied van de grafentheorie en de statistische mechanica, waartoe o.a. een tweetal beurzen van de Nederlandse Organisatie voor Zuiver Wetenschappelijk Onderzoek (Z.W.O.) mij in staat stelde. Bijzonder stimulerend voor het onderzoek waarop dit proefschrift betrekking heeft, was mijn deelname aan de Enrico Fermi-Zomerschool die gehouden werd in juli/augustus 1970 te Varenna, Italië over Kritische Verschijnselen.

Behalve bij het onderzoek ben ik op verschillende manieren betrokken geweest bij het onderwijs voor doctoraalstudenten in de natuurkunde.

Sedert 16 februari 1973 ben ik werkzaam bij het Centraal Bureau voor de Statistiek te 's-Gravenhage op de hoofdafdeling statistische analyse.

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