Answers to the Exam Quantum Information, 20 December 2024

each item gives 2 points for a fully correct answer, grade = total $\times 9/24 + 1$

1. *(a)* the first qubit is 0 with probability $1/4 + 1/4 = 1/2$.
   (b) $\rho = |\phi\rangle\langle\phi|$ with $|\phi\rangle = \sqrt{\frac{1}{2}}(|0\rangle + |1\rangle)$, or $\rho = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$;
   this is the pure state $|\phi\rangle$, and indeed $\rho^2 = \rho$.
   (c) the concurrence equals $C = 2|\frac{1}{2\sqrt{6}} + \frac{i}{2\sqrt{3}}| \neq 0$, so the state is entangled.

2. *(a)* CNOT operation with the first qubit as the control and the second qubit as the target, followed by the bit flip (X) of the second qubit.
   (b) the no-cloning theorem forbids copying an unknown state, but the copy of $|\psi_1\rangle$ would have been $(\alpha|0\rangle + \beta|1\rangle)(\alpha|0\rangle + \beta|1\rangle)$, which is different from $\alpha|0\rangle|0\rangle + \beta|1\rangle|1\rangle$, so the no-cloning theorem is not violated.
   (c) suppose there is a unitary U such that $|\psi\rangle|0\rangle \mapsto |\psi\rangle|\psi\rangle$; then $|0\rangle|0\rangle \mapsto |0\rangle|0\rangle$ and $|1\rangle|0\rangle \mapsto |1\rangle|1\rangle$, hence $(|0\rangle + |1\rangle)|0\rangle \mapsto |0\rangle|0\rangle + |1\rangle|1\rangle$, which is different from the copied state $(|0\rangle + |1\rangle)(|0\rangle + |1\rangle)$.

3. *(a)* The σ_x maps the state $|0\rangle + |1\rangle$ onto itself, so it is not a NOT operation.
   (b) The state $|\psi'\rangle = \beta^*|0\rangle - \alpha^*|1\rangle$ is orthogonal to $|\psi\rangle$.
   (c) The NOT operation involves a complex conjugation, so it is not unitary, hence not a valid quantum gate.

4. *a)* If $f(0) = f(1)$ the function $f(x)$ is not invertible, so the single-qubit gate $|x\rangle \mapsto |f(x)\rangle$ would not be a unitary operation.
   b) $|\Psi_1\rangle = |0\rangle|1\rangle$, $|\Psi_2\rangle = (|0\rangle + |1\rangle)(|0\rangle - |1\rangle)$, $|\Psi_3\rangle = |0\rangle(|f(0)\rangle - |1 \oplus f(0)\rangle) + |1\rangle(|f(1)\rangle - |1 \oplus f(1)\rangle)$, $|\Psi_4\rangle = (|0\rangle + |1\rangle)(|f(0)\rangle - |1 \oplus f(0)\rangle) + (|0\rangle - |1\rangle)(|f(1)\rangle - |1 \oplus f(1)\rangle)$.
   c) if $f(0) = f(1)$ the state $|\Psi_4\rangle \propto |0\rangle(|f(0)\rangle - |1 \oplus f(0)\rangle)$ so a measurement of the first qubit will always give the value 0; while if $f(0) \neq f(1)$ the state $|\Psi_4\rangle \propto |1\rangle(|f(0)\rangle - |1 \oplus f(0)\rangle)$, so a measurement of the first qubit will always give the value 1.