Exam Quantum Information, 20 December 2024, 13–16 hours.

1. Consider the following two-qubit state

$$|\psi\rangle = \sqrt{\tfrac{1}{4}}|00\rangle + i\sqrt{\tfrac{1}{3}}|10\rangle + \sqrt{\tfrac{1}{4}}|01\rangle - \sqrt{\tfrac{1}{6}}|11\rangle.$$

The first qubit is measured and the outcome is 0.

- *a)* What is the probability for this outcome to happen?
- *b)* Which density matrix describes the state of the second qubit after this measurement? Is it a pure state or a mixed state? Motivate your answer.
- *c)* Are the two qubits in the original state $|\psi\rangle$ entangled or not? Motivate your answer.

2. Bob has two qubits. The first qubit is in the *unknown* state $|\psi_1\rangle = \alpha|0\rangle + \beta|1\rangle$, the second qubit is in the known state $|\psi_2\rangle = |1\rangle$.

- *a)* Can you construct a quantum operation that transforms $|\psi_1\rangle|\psi_2\rangle$ into $\alpha|0\rangle|0\rangle + \beta|1\rangle|1\rangle$?
- *b)* Bob says such an operation cannot be a unitary operation because it would violate the "no-cloning theorem". What is your answer to this claim?
- *c)* Give a proof of the no-cloning theorem.

3. A NOT operation is defined as an operation that transforms an unknown state $|\psi\rangle$ into a state $|\psi'\rangle$ that is orthogonal to $|\psi\rangle$.

- *a)* Alice says that the X gate (which exchanges the states $|0\rangle$ and $|1\rangle$) is an example of a single-qubit NOT operation. Do you agree?
- *b)* Consider the state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, with arbitrary complex numbers α, β. Perform the NOT operation on the state and give the resulting state $|\psi'\rangle$.
- *c)* Eve would like to implement the NOT operation as a gate on a quantum computer. Is this possible?

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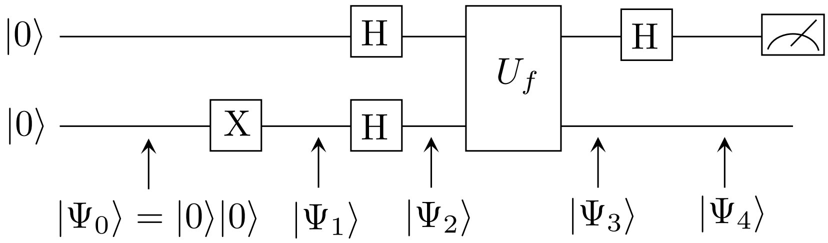
4. The Deutsch algorithm can tell whether a function f from $\{0, 1\}$ to $\{0, 1\}$ satisfies $f(0) = f(1)$ or $f(0) \neq f(1)$. It does so with a *single* evaluation of f in a two-qubit gate U_f that maps $|x\rangle|y\rangle \mapsto |x\rangle|y \oplus f(x)\rangle$.

- *a)* Why is it in general not possible to represent the operation $|x\rangle \mapsto |f(x)\rangle$ by a single-qubit gate?

  The diagram below shows the circuit, containing in addition to the gate U_f three single-qubit Hadamard gates H and a Pauli gate $X = \sigma_x$. The top line is the qubit $|x\rangle$, the bottom line the qubit $|y\rangle$.



- *b)* Give the expressions for the two-qubit states $|\Psi_n\rangle$ at each stage $n = 1, 2, 3, 4$ of the quantum computation.
- *c)* After the final Hadamard gate that qubit is measured. Explain how the measurement outcome decides whether $f(0) = f(1)$ or $f(0) \neq f(1)$.