Quantum Information: lecture 1

- quantum bits — Preskill 2.1 and 2.2
- quantum gates — Preskill 6.2
- density matrix — Preskill 2.3

in this lecture, mainly a quantum primer

quantum states, Hilbert space, Hermitian and unitary operators, Schrödinger and Heisenberg equations, pure and mixed states

for a linear algebra reminder, see separate notes

Quantum primer 1: states & operators

state: complex N-dimensional vector *Hilbert space* ($N = 2$ qubit)
row vector $\langle\psi|$, column vector $|\psi\rangle$ *bra-ket*
scalar product: $\langle\chi|\phi\rangle = \langle\phi|\chi\rangle^*$ normalization: $\langle\psi|\psi\rangle = 1$
only absolute value $|\langle\chi|\phi\rangle|$ is observable
$|\psi\rangle$ and $e^{i\alpha}|\psi\rangle$ describe *the same* state

operator: complex $N \times N$ matrix
$|\phi\rangle \mapsto A|\phi\rangle$, $\langle\chi|\phi\rangle \mapsto \langle\chi|A|\phi\rangle = \langle\phi|A^\dagger|\chi\rangle^*$
Hermitian conjugate or adjoint operator: $(A^\dagger)_{nm} = A^*_{mn}$
self-adjoint (*Hermitian*): $A^\dagger = A$ (real eigenvalues, observable)

change of basis: $|\phi\rangle \mapsto U|\phi\rangle$, $|\chi\rangle \mapsto U|\chi\rangle$, $\langle\chi|\phi\rangle \mapsto \langle\chi|U^\dagger U|\phi\rangle$,
invariant if $U^\dagger U = 1$ or $U^{-1} = U^\dagger$ *unitary operator*
examples: $U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ *Hadamard gate*, $U = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ *Pauli Z-gate*

Quantum primer 2: time evolution

states evolve in time according to the Schrödinger equation:

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = H|\psi(t)\rangle \Rightarrow |\psi(t)\rangle = e^{-iHt/\hbar}|\psi(0)\rangle$$

the time evolution is a *unitary transformation*

$\Rightarrow$ quantum computations must be *reversible* — AND, OR operations forbidden, CNOT allowed

Schrödinger picture: time dependence in states

Heisenberg picture: time dependence in operators

$$\langle\phi(t)|O|\chi(t)\rangle = \langle\phi|O(t)|\chi\rangle$$

with $|\psi\rangle \equiv |\psi(0)\rangle$ and $O(t) = e^{iHt/\hbar}Oe^{-iHt/\hbar}$

$$i\hbar\frac{d}{dt}O = OH - HO \equiv [O, H] \quad \text{(commutator)}$$

Heisenberg equation of motion

Pure & mixed states, density matrix

state $|\psi\rangle$ corresponds to operator $\rho = |\psi\rangle\langle\psi|$ (density matrix)

ρ acting on a state produces a new state:
$\rho|\phi\rangle = \text{constant} \times |\psi\rangle$, where constant $= \langle\psi|\phi\rangle$

$$\text{check that:}\ \ \mathrm{Tr}\,\rho = 1,\quad \rho = \rho^\dagger,\quad \rho^2 = \rho.$$

expectation value of operator M in state $|\psi\rangle$:

$$\langle M\rangle = \langle\psi|M|\psi\rangle = \mathrm{Tr}\,M\rho.$$

why bother? because the density matrix is more general:

$$\rho = \sum_n p_n|\psi_n\rangle\langle\psi_n|,\ \ 0 \leqslant p_n \leqslant 1,\ \ \sum_n p_n = 1.$$

mixed state: mixture of pure states $|\psi_n\rangle$, appearing with weight p_n.

$$\text{check that still:}\ \ \mathrm{Tr}\,\rho = 1,\quad \rho = \rho^\dagger,\quad \text{but}\ \ \rho^2 \neq \rho.$$

Evolution of the density matrix

Schrödinger equation: $i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = H|\psi(t)\rangle$

$$\Rightarrow i\hbar\frac{\partial}{\partial t}\rho(t) = -[\rho(t), H].$$

note *minus* sign compared to Heisenberg equation.

solution: $\rho(t) = e^{-iHt/\hbar}\rho(0)e^{iHt/\hbar}.$

Implication: pure state at $t = 0$ remains pure for $t > 0$.

Q: in the laboratory we typically find mixed states, does this imply that the Universe was not in a pure state at the Big Bang?

A: no, partial observation converts a pure state into a mixed state

$|\psi\rangle = \frac{1}{\sqrt{2}}|\uparrow_A\uparrow_B\rangle + \frac{1}{\sqrt{2}}|\downarrow_A\downarrow_B\rangle,$ entangled pure state

$\text{Tr}_B|\psi\rangle\langle\psi| = \frac{1}{2}|\uparrow_A\rangle\langle\uparrow_A| + \frac{1}{2}|\downarrow_A\rangle\langle\downarrow_A|$ partial trace