Answers to the Exam Quantum Theory, 22 december 2014

each item gives 2 points for a fully correct answer, grade = total $\times\, 9/24 + 1$

1. *a)* $\bar{A} = \langle\Psi|A|\Psi\rangle = \langle e^{-iH_0t/\hbar}\Psi_{\rm I}|e^{-iH_0t/\hbar}A_{\rm I}e^{iH_0t/\hbar}|e^{-iH_0t/\hbar}\Psi_{\rm I}\rangle = \langle\Psi_{\rm I}|A_{\rm I}|\Psi_{\rm I}\rangle$.
   b) $dA_{\rm I}/dt = (i/\hbar)H_0A_{\rm I} - (i/\hbar)A_{\rm I}H_0 = (i/\hbar)[H_0, A_{\rm I}]$.
   c) $i\hbar d\Psi/dt = H\Psi \to i\hbar d\Psi_{\rm I}/dt = -H_0\Psi_{\rm I} + e^{iH_0t/\hbar}H\Psi = (-H_0 + H_{\rm I})\Psi_{\rm I} = V_{\rm I}\Psi_{\rm I}$.

2. *a)* $\langle\tilde{\Psi}|\tilde{\Psi}\rangle = \langle\tilde{\Psi}_\uparrow|\tilde{\Psi}_\uparrow\rangle + \langle\tilde{\Psi}_\downarrow|\tilde{\Psi}_\downarrow\rangle = \langle\Psi_\downarrow|\Psi_\downarrow\rangle^* + \langle\Psi_\uparrow|\Psi_\uparrow\rangle^* = \langle\Psi|\Psi\rangle^* = 1$; application of the operation twice gives $(\Psi_\uparrow, \Psi_\downarrow) \mapsto (\Psi_\downarrow^*, -\Psi_\uparrow^*) \mapsto (-\Psi_\uparrow, -\Psi_\downarrow)$.
   b) application of H to $(\Psi_\downarrow^*, -\Psi_\uparrow^*)$ gives for the first component $(p^2/2m + V)\Psi_\downarrow^* - (\alpha p_x - i\alpha p_y)\Psi_\uparrow^* = [(p^2/2m + V)\Psi_\downarrow + (\alpha p_x + i\alpha p_y)\Psi_\uparrow]^* = E\Psi_\downarrow^*$. (Note that $p_x^* = -p_x$.) Similarly, for the second component we find $-E\Psi_\uparrow^*$. So $(\Psi_\downarrow^*, -\Psi_\uparrow^*)$ is an eigenstate of H at eigenvalue E.
   c) Suppose the two eigenstates are linearly related, then there is a complex coefficient $\lambda \neq 0$ such that $\tilde{\Psi}_\uparrow = \lambda\Psi_\uparrow$ and $\tilde{\Psi}_\downarrow = \lambda\Psi_\downarrow$. This requires that $\Psi_\downarrow^* = \lambda\Psi_\uparrow$ and $-\Psi_\uparrow = \lambda^*\Psi_\downarrow^*$, hence $-\Psi_\uparrow = \lambda^*\lambda\Psi_\uparrow$, which is only possible if $\Psi_\uparrow = 0$. But then also $\Psi_\downarrow = 0$, which is not possible because of the normalization. *Alternatively,* show that $\langle\Psi|\tilde{\Psi}\rangle = \Psi_\uparrow^*\tilde{\Psi}_\uparrow + \Psi_\downarrow^*\tilde{\Psi}_\downarrow = \Psi_\uparrow^*\Psi_\downarrow^* - \Psi_\downarrow^*\Psi_\uparrow^* = 0$.

3. *a)* $\langle\alpha|\beta\rangle = e^{-|\alpha|^2/2}\langle 0|e^{\alpha^* a}|\beta\rangle = e^{-|\alpha|^2/2}e^{\alpha^*\beta}\langle 0|\beta\rangle = e^{-|\alpha|^2/2-|\beta|^2/2}e^{\alpha^*\beta}$, hence $|\langle\alpha|\beta\rangle|^2 = e^{-|\alpha|^2-|\beta|^2}e^{\alpha^*\beta+\alpha\beta^*} = e^{-|\alpha-\beta|^2}$.
   b) $\bar{n} = \langle\beta|a^\dagger a|\beta\rangle = \beta^*\beta$, $\overline{n^2} = \langle\beta|(a^\dagger a)^2|\beta\rangle = \langle\beta|(a^\dagger)^2a^2|\beta\rangle + \langle\beta|a^\dagger a|\beta\rangle = (\beta^*\beta)^2 + \beta^*\beta$, so $\operatorname{var} n = \beta^*\beta = \bar{n}$.
   c) $\bar{n} = \operatorname{Tr}\rho a^\dagger a = p|\alpha|^2 + (1-p)|\beta|^2$, $\overline{n^2} = \operatorname{Tr}\rho(a^\dagger a)^2 = p(|\alpha|^4 + |\alpha|^2) + (1-p)(|\beta|^4 + |\beta|^2)$, $\operatorname{var} n = p(1-p)(|\alpha|^2 - |\beta|^2)^2 + \bar{n} > \bar{n}$.

4. *a)* The wave function decays to zero for $x \to \pm\infty$, it is symmetric without a node for the ground state, it is antisymmetric with one node at $x = 0$ for the first excited state, and symmetric with two nodes for the second excited state.
   b) There are two turning points where the velocity goes to zero smoothly, and these are associated with a phase shift of $-\pi/2$, so the total phase shift is $\gamma = -\pi$.
   c) $E = p_x^2/2m + V_0|x| \to p_x = \pm\sqrt{2m(E - V_0|x|)}$;
   $\oint p_x dx = 4\int_0^{E/V_0}\sqrt{2m(E - V_0x)}dx = \frac{8}{3}(2m)^{1/2}V_0^{-1}E^{3/2} = 2\pi\hbar(n + 1/2)$;
   $E_n = (3\pi\hbar V_0/4)^{2/3}(2m)^{-1/3}(n + 1/2)^{2/3}$.