Answers to the Exam Quantum Theory, retake, 12 January 2014
each item gives 2 points for a fully correct answer, grade = total $\times 9/24 + 1$

1. *a)* $\langle\psi|\rho\psi\rangle = \sum_n p_n |\langle\psi|\psi_n\rangle|^2 \geq 0$.
   b) $d\rho/dt = \sum_n p_n(|d\Psi_n/dt\rangle\langle\Psi_n| + |\Psi_n\rangle\langle d\Psi_n/dt|) = (-i/\hbar)\sum_n p_n(H|\Psi_n\rangle\langle\Psi_n| - |\Psi_n\rangle\langle\Psi_n|H) = (-i/\hbar)[H,\rho]$.
   c) $\rho(t) = e^{-iHt/\hbar}\rho(0)e^{iHt/\hbar}$ so $\rho^2(t) - \rho(t) = e^{-iHt/\hbar}[\rho^2(0) - \rho(0)]e^{iHt/\hbar}$ and $\rho^2(0) - \rho(0) = e^{iHt/\hbar}[\rho^2(t) - \rho(t)]e^{-iHt/\hbar}$; hence $\rho^2(t) = \rho(t) \Leftrightarrow \rho^2(0) = \rho(0)$.

2. *a)* $dE_n/d\lambda = (d/d\lambda)\langle n,\lambda|H(\lambda)|n,\lambda\rangle = \langle n,\lambda|\partial H/\partial\lambda|n,\lambda\rangle$ plus a term consisting of the integral $\int[(H\psi)^*\partial\psi/\partial\lambda + (\partial\psi^*/\partial\lambda)H\psi]dx$, with ψ the eigenstate of H. Because $H\psi = E_n(\lambda)\psi$, this integral can also be written as $E_n(\lambda)\int[\psi^*\partial\psi/\partial\lambda + (\partial\psi^*/\partial\lambda)\psi]dx = E_n(\lambda)(d/d\lambda)\int|\psi|^2\,dx = 0$ because of the normalization of ψ.
   b) the operator $p_z = -i\hbar\partial/\partial z$ commutes with H, so the Heisenberg equation of motion gives $dp_z/dt = 0$. The velocity operator $v_z = (p_z - eBy)/m$ does not commute with H, so the velocity along the wire is not conserved.
   c) the velocity operator in the z-direction is $v_z = \partial H/\partial p_z$; now use the Hellman-Feynman theorem, $\langle v_z\rangle = dE(p_z)/dp_z$.

3. *a)* the flux is $\Phi = \pi|B|R_c^2 = \pi|B|(2mE)/(qB)^2$, so quantization gives $E_n = \hbar(|qB|/m)(n+\frac{1}{2})$. The ground state is $E_0 = |qB| \times (\hbar/2m)$.
   b) In graphene $E_n = \text{constant} \times \sqrt{n|B|}$, so it increases more slowly with $|B|$ and is zero for $n = 0$.
   c) motion along z is independent of motion in x-y plane, adds $p_z^2/2m = (\hbar k)^2/2m$ to the energy; $E_n(k) = (n+1/2)\hbar|qB|/m + \hbar^2k^2/2m$, so spacing is $\hbar|qB|/m$.

4. *a)* The trace of $H(t)$ is zero, so the eigenvalues are $\pm E$, and the determinant is $-(\hbar e/2m)^2B_0^2 = -E^2$, so the eigenvalues are $\pm\hbar eB_0/2m = \pm\frac{1}{2}\hbar\omega_0$.
   b) The state $\psi(0)$ is an eigenstate of $H(0)$ with eigenvalue $E = -\frac{1}{2}\hbar\omega_0$, so the dynamical phase is $c_1T = -iET/\hbar$ with $c_1 = \frac{1}{2}\omega_0$.
   c) The normalized and single-valued eigenstate of $H(t)$ with eigenvalue $E = -\frac{1}{2}\hbar\omega_0$ is $|t\rangle = 2^{-1/2}(1, e^{2\pi it/T})$. The Berry phase is given by
   $c_0 = i\int_0^T dt\,\langle t|\frac{d}{dt}|t\rangle = i\int_0^T dt\,\frac{1}{2}(2\pi i/T)\begin{pmatrix}1\\ e^{-2\pi it/T}\end{pmatrix}\cdot\begin{pmatrix}0\\ e^{2\pi it/T}\end{pmatrix} = -\pi$.