Answers to the Exam Quantum Theory, 21 december 2015

each item gives 2 points for a fully correct answer, grade = total $\times\, 9/24 + 1$

1. *a)* a density matrix should be Hermitian, it should have trace 1, and all its eigenvalues should be ≥ 0.
   b) $\hat{\rho}_1$ has a negative eigenvalue $-1/3$, $\hat{\rho}_2$ is not Hermitian, so neither is a density matrix; $\hat{\rho}_3$ is Hermitian, has trace 1, and eigenvalues 0 and 1, so it is a density matrix.
   c) a pure state has $\hat{\rho}^2 = \hat{\rho}$, only $\hat{\rho}_4$ satisfies: it is the density matrix $|\psi\rangle\langle\psi|$ of the pure state $|\psi\rangle = 2^{-1/2}(|0\rangle + |1\rangle)$.

2. *a)* $\hat{S}^\dagger(r) = \hat{S}(-r) = \hat{S}^{-1}(r)$.
   b) $\langle r|\hat{a}|r\rangle = 0$, $\langle r|\hat{a}^2|r\rangle = -\frac{1}{2}\sinh 2r = \langle r|(\hat{a}^\dagger)^2|r\rangle$, $\langle r|\hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a}|r\rangle = \cosh 2r$, so $\langle r|(\hat{a}^2 + (\hat{a}^\dagger)^2\rangle = \cosh 2r - \sinh 2r = e^{-2r}$.
   c) the Heisenberg uncertainty principle says that $\operatorname{Var} x \times \operatorname{Var} p \geq 1/4$, here the minimum uncertainty is reached; for large positive r the uncertainty in position is much smaller than in momentum (hence the name "squeezed" state).

3. *a)* the N-th level has $\oint p\,dx = N \times 2\pi\hbar$, with $E = V(x) + p^2/2m$; the integral extends over one period of the motion, so from a to b and back to a; this is a good approximation for $N \gg 1$.
   b) $dN/dE = (m/\pi\hbar)\int_a^b p^{-1}dx = (\pi\hbar)^{-1}\int_a^b v^{-1}dx = T/\pi\hbar$.
   c) There are two soft turning points, so $\gamma = 2\times\pi/2$, and the Bohr-Sommerfeld quantization rule is $\oint p\,dx = (n + 1/2)2\pi\hbar$; the turning points for $V(x) = cx^2$ are at $\pm\sqrt{E/c}$; for the lowest level ($n = 0$) we have $\pi\hbar = 4(2m)^{1/2}\int_0^{\sqrt{E/c}}(E - cx^2)^{1/2}\,dx = 4(2mE)^{1/2} \times (E/c)^{1/2} \times \int_0^1 \sqrt{1 - x^2}\,dx = \pi E(2m/c)^{1/2}$, so the energy of the lowest level is $E = \hbar(c/2m)^{1/2}$.

4. *a)* $d\vec{r}/dt = (i/\hbar)[H, \vec{r}] = (1/m)(\vec{p} - q\vec{A})$
   b) $\exp(iq\chi/\hbar)(\vec{p} - q\vec{A})^2\exp(-iq\chi/\hbar) = [\exp(iq\chi/\hbar)(\vec{p} - q\vec{A})\exp(-iq\chi/\hbar)]^2 = (\vec{p} - q\vec{A} - q\nabla\chi)^2 = (\vec{p} - q\vec{A}')^2$
   c) H' is related to H by a unitary transformation, so it represents the same system described in a different basis.