Answers to the Exam Quantum Theory, 20 January 2016

each item gives 2 points for a fully correct answer, grade = total $\times\, 9/24 + 1$

1. *a)* $\psi(p) = (2\pi\hbar)^{-1/2} \int_{-\infty} e^{-ipx/\hbar}\psi(x)dx$,

$$\int_{-\infty}^{\infty} |\psi(p)|^2 dp = (2\pi\hbar)^{-1} \int dp \int dx \int dx' e^{ip(x'-x)/\hbar}\psi(x)\psi^*(x')$$

$$= \int dx \int dx' \delta(x - x')\psi(x)\psi^*(x') = \int dx |\psi(x)|^2 = 1.$$

b)

$$\mathcal{T}\psi(p) = (2\pi\hbar)^{-1/2} \int_{-\infty} e^{-ipx/\hbar}\mathcal{T}\psi(x)dx = (2\pi\hbar)^{-1/2} \int_{-\infty} e^{-ipx/\hbar}\psi^*(x)dx$$

$$= (2\pi\hbar)^{-1/2} \left(\int_{-\infty} e^{ipx/\hbar}\psi(x)dx\right)^* = \psi^*(-p).$$

c) Kramers theorem requires that the time-reversal symmetry operator squares to -1, here $\mathcal{T}^2 = +1$ so it does not hold.

2. *a)* The operators commute for bosons and anticommute for fermions, to ensure that the wave function is symmetric under particle exchange for bosons and antisymmetric for fermions.
b) use $c_i c_j^\dagger = \pm c_j^\dagger c_i + \delta_{ij}$, with the plus sign for bosons and the minus sign for fermions; $\langle 0|c_\alpha c_\beta c_\alpha^\dagger c_\beta^\dagger|0\rangle = \pm\langle 0|c_\alpha c_\alpha^\dagger c_\beta c_\beta^\dagger|0\rangle = \pm 1$, because $c_i|0\rangle = 0$.
c) $a_i a_j^\dagger + a_j^\dagger a_i = \sum_{kl} U_{ik}U_{jl}^*(c_k c_l^\dagger + c_l^\dagger c_k) = \sum_{kl} U_{ik}U_{jl}^*\delta_{kl} = \delta_{ij}$.

3. *a)* $\langle\psi|\psi\rangle = \sum_{n,m} c_n^* c_m \langle\Phi_n|\Phi_m\rangle = \sum_n |c_n|^2$, because $\langle\Phi_n|\Phi_m\rangle = \delta_{nm}$.
b) $\langle\psi|H-E_0|\psi\rangle = \sum_{n,m} c_n^* c_m \langle\Phi_n|H-E_0|\Phi_m\rangle = \sum_{n,m} c_n^* c_m (E_m-E_0)\langle\Phi_n|\Phi_m\rangle = \sum_n |c_n|^2(E_n - E_0) \geq 0$.
c) Calculate $E(a) = \int_{-\infty}^{\infty} \Phi_a^*(x)H\Phi_a(x)dx$, and solve $dE(a)/da = 0$ to find the minimal value $E_{\min}$ of $E(a)$ as a function of $a > 0$. This is the optimal upper bound of the ground state energy E_0.

4. *a)* Insert a resolution of the identity, $\mathbb{1} = \sum_n |n\rangle\langle n|$, to arrive at $\int dx\, G(x,x;t) = \int dx \sum_n \langle x|e^{-iHt/\hbar}|n\rangle\langle n|x\rangle = \sum_n e^{-iE_n t/\hbar} \int dx\, |\langle x|n\rangle|^2 = \sum_n e^{-iE_n t/\hbar}$.
b) $S[x(t')] = \int_0^t L[x(t'), \dot{x}(t')]dt'$, with $L(x,\dot{x}) = p\dot{x} - H = \dot{x}^2/2m - V(x)$ the Lagrangian.
c) The Fourier transform $F(t)$ of $\rho(E)$ contains only closed paths, and in the semiclassical limit $\hbar \to 0$ only classical paths contribute.