Answers to the Exam Quantum Theory, 5 January 2018

each item gives 2 points for a fully correct answer, grade = total $\times\, 9/24 + 1$

1. *(a)* $\rho^2 \neq \rho$, so the particle is in a mixed state.
   (b) $\langle S_z\rangle = \mathrm{Tr}\, S_z\rho = 1/4$
   (c) $\rho = |0\rangle\langle 0| = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $\rho^2 = \rho$, so the particle is in a pure state.

2. *(a)* The difference between $H = (\boldsymbol{p} - e\boldsymbol{A})^2/2m$ and $H' = (\boldsymbol{p} - e\boldsymbol{A}')^2/2m$ is a different choice of vector potential, $\boldsymbol{A} = B(0, x, 0)$ and $\boldsymbol{A}' = (B/2)(-y, x, 0)$; both vector potentials give the same magnetic field $\boldsymbol{B} = \nabla \times \boldsymbol{A}$ so the energy spectrum of H and H' must be the same; another way to see this, is to note that H and H' are related by a unitary transformation: $H' = e^{-ixyeB/2\hbar} H e^{ixyeB/2\hbar}$.
   (b) $[a, a^\dagger] = (2e\hbar B)^{-1} 2[p_x, ieBx] = 1$, since $p_x = -i\hbar\partial/\partial x$.
   (c) $a^\dagger a = (2e\hbar B)^{-1}(p_x^2 + (p_y - eBx)^2 - ieB[p_x, x]) = (m/e\hbar B)H - 1/2$. This is the Hamiltonian of a harmonic oscillator with frequency $\omega = eB/m$, so the energy levels (Landau levels) are at $E_n = (n + 1/2)\hbar\omega$, $n = 0, 1, 2, \ldots$.

3. *(a)* The momentum $p = mv + eA$ in the Bohr-Sommerfeld rule is the canonical momentum, not just the mechanical momentum.
   (b) The contribution to $\oint p \cdot dq$ from the electromagnetic momentum is $e\oint A \cdot dq = -eB\pi l_{\text{cycl}}^2 = -2\pi mE/eB$, which gives $E_n = \hbar\omega_n(n + 1/2)$. It differs from Nicandro's answer by a factor two.
   (c) For massless electrons we have $E = pv$, $l_{\text{cycl}} = p/eB$, so $\oint p \cdot dq = p \times 2\pi l_{\text{cycl}} - eB \times \pi l_{\text{cycl}}^2 = \pi p^2/eB = \pi E^2/(eBv^2)$; the quantization is $E_n^2 = 2\hbar eBv^2(n + \gamma)$; the offset $\gamma = 0$ because the phase shift from the turning points is canceled by the Berry phase.

4. *(a)* $U^\dagger = e^{(it/\hbar)H} = U^{-1}$; $i\hbar\partial\psi/\partial t = HU(t)\psi(0) = H\psi(t)$.
   (b) insert the resolution of the identity: $G(x, x_0; t) = \int dp\, \langle x|p\rangle\langle p|U(t)|x_0\rangle = \int dp \langle x|p\rangle\langle p|x_0\rangle U(t)$ and then substitute $\langle x|p\rangle = (2\pi\hbar)^{-1/2} e^{ipx/\hbar}$;
   $G(x, x_0; 0) = \delta(x - x_0)$, because $\delta(x) = (2\pi)^{-1}\int dk\, e^{ikx}$.
   (c) The exponent $m(x - x_0)^2/2t$ equals the action along the classical path, $S_{\text{classical}} = tp^2/2m = (x - x_0)^2 m/2t$, so $G \propto \exp(iS_{\text{classical}}/\hbar)$. This is the dominant contribution in the $\hbar \to 0$ limit. Fluctuations around the classical path give the prefactor of the propagator.