ANSWERS TO THE EXAM QUANTUM THEORY, 16 JANUARY 2019

each item gives 2 points for a fully correct answer, grade = total $\times\, 9/24 + 1$

1. *(a)* $\langle\Phi|\Phi\rangle = \left\langle\psi\,\middle|\,A^2 + \frac{\hbar^2}{4(\Delta B^2)^2}B^2 + \frac{i\hbar}{2\Delta B^2}[A,B]\,\middle|\,\psi\right\rangle$
   $= \Delta A^2 + \frac{\hbar^2}{4\Delta B^2} - \frac{\hbar^2}{2\Delta B^2} = \Delta A^2 - \frac{\hbar^2}{4\Delta B^2}$,
   and since $\langle\Phi|\Phi\rangle \geq 0$ we conclude that $\Delta A^2 \Delta B^2 \geq \hbar^2/4$.
   (b) Expand $e^{i\omega\tau}$ in a Taylor series,
   $$[H, e^{i\omega\tau}] = \sum_{p=0}^{\infty}[H, \frac{1}{p!}(i\omega\tau)^p] = -\hbar\omega\sum_{p=1}^{\infty}\frac{1}{(p-1)!}(i\omega\tau)^{p-1} = -\hbar\omega e^{i\omega\tau}.$$
   (c) $H\psi' = He^{i\omega\tau}\psi_E = e^{i\omega\tau}H\psi_E + [H, e^{i\omega\tau}]\psi_E = e^{i\omega\tau}E\psi_E - \hbar\omega e^{i\omega\tau}\psi_E = (E - \hbar\omega)\psi'$.

2. *(a)* Since $[N,H] \neq 0$ the particle number is not conserved, the pair potential can change it between 0 and 2.
   (b) The matrix elements between states that differ by 1 particle vanish, because H conserves the parity of the particle number. The full Hamiltonian is $H = \begin{pmatrix} 0 & \Delta^* & 0 & 0 \\ \Delta & -2\mu & 0 & 0 \\ 0 & 0 & -\mu & 0 \\ 0 & 0 & 0 & -\mu \end{pmatrix}$
   (c) For $\Delta = 0$, the spectrum of H consists of an empty state with energy 0, a two-fold degenerate singly-occupied state with energy $-\mu$, and doubly-occupied state with energy -2μ.With superconductivity the energy of the two singly-occupied states is unchanged, the other two states have energies $-\mu \pm \sqrt{\mu^2 + |\Delta|^2}$.

3. *(a)* The amplitude of the transmitted wave is
   $$\Psi_{\text{out}} = \Psi_{\text{in}}T\left[1 + Re^{i\phi} + (Re^{i\phi})^2 + (Re^{i\phi})^3 + \cdots\right] = \frac{T\Psi_{\text{in}}}{1 - Re^{i\phi}}.$$
   The absolute value squared of $\Psi_{\text{out}}/\Psi_{\text{in}}$ then gives $P = T^2(1+R^2-2R\cos\phi)^{-1}$.
   (b) Constructive interference of the waves for $\phi = 0$ gives resonant transmission with unit probability.
   (c) For an enclosed flux $\Phi = BL^2$ the electron picks up an Aharonov-Bohm phase $2\pi\Phi e/h$ which adds to ϕ, so the transmission probability oscillates between $T^2/(1+R)^2$ and 1 with period $\Delta\Phi = h/e \Rightarrow \Delta B = L^{-2}h/e$.

4. *a)* The wave function decays to zero for $x \to \pm\infty$, it is symmetric without a node for the ground state, it is antisymmetric with one node at $x = 0$ for the first excited state, and symmetric with two nodes for the second excited state.
   b) There are two turning points where the velocity goes to zero smoothly, and these are associated with a phase shift of $-\pi/2$, so the total phase shift is $\gamma = -\pi$.

c) $E = p_x^2/2m + V_0 x^2 \to p_x = \pm\sqrt{2m(E - V_0 x^2)}$;
$\oint p_x dx = 4\int_0^{\sqrt{E/V_0}} \sqrt{2m(E - V_0 x^2)}dx = 4(2mE^2/V_0)^{1/2}\int_0^1 \sqrt{1 - x^2}dx$
$= \pi E(2m/V_0)^{1/2} = 2\pi\hbar(n + 1/2)$;
$\Rightarrow E_n = \hbar(2V_0/m)^{1/2}(n + 1/2)$.