ANSWERS TO THE EXAM QUANTUM THEORY, 25 JANUARY 2021

each item gives 2 points for a fully correct answer, grade = total $\times 9/24 + 1$

1. *(a)* P is Hermitian if $P = P^\dagger$, with the Hermitian conjugate $P^\dagger$ defined by $\langle\phi|P\psi\rangle = \langle P^\dagger\phi|\psi\rangle$; the parity operator is Hermitian because

$$\langle\phi|P\psi\rangle = \int dx\, \phi^*(x)P\psi(x) = \int dx\, \phi^*(x)\psi(-x) = \int dx\, \phi^*(-x)\psi(x) = \langle P^\dagger\phi|\psi\rangle$$

   (b) $P^2 = I$ equals the identity operator I, so $P^{-1} = P = P^\dagger$, which is the definition of a unitary operator; the eigenvalues satisfy $p^2 = 1 \to p = \pm 1$.
   (c) If ψ is an eigenstate of H with a nondegenerate eigenvalue E, and P commutes with H, then also $P\psi$ is an eigenstate of H with the same eigenvalue E, since $HP\psi = PH\psi = EP\psi$; if E is nondegenerate, the two eigenstates ψ and $P\psi$ must be linearly related, so we must have $P\psi = \lambda\psi$ for some number λ, hence ψ is an eigenfunction of the parity operator, hence either $\lambda = +1$ and $\psi(x) = \psi(-x)$ (even function) or $\lambda = -1$ and $\psi(x) = -\psi(-x)$ (odd function).

2. *a)* $S^\dagger(s) = \exp\left(\frac{1}{2}sa^\dagger a^\dagger - \frac{1}{2}saa\right) = S(s)^{-1} \neq S(s)$, so this operator is unitary but not Hermitian.
   b) $\langle s|\hat{x}|s\rangle = \langle 0|S^\dagger(s)\hat{x}S(s)|0\rangle$
   $= 2^{-1/2}\langle 0|(a\cosh s - a^\dagger \sinh s) + (a^\dagger\cosh s - a\sinh s)|0\rangle = 0$
   $\langle s|\hat{x}^2|s\rangle = (1/2)\langle 0|\big(a(\cosh s - \sinh s) + a^\dagger(\cosh s - \sinh s)\big)^2|0\rangle$
   $= (1/2)(\cosh s - \sinh s)^2 = (1/2)e^{-2s}$.
   c) there is no contradiction: the uncertainty principle provides a lower bound to the product of the variance of position and momentum, so if the variance of position goes to zero for $s \to 0$, the variance of momentum must diverge as e^{2s}.

3. *a)* $H = -(\mu B_0/2)\begin{pmatrix} 0 & e^{-i\omega t} \\ e^{i\omega t} & 0 \end{pmatrix}$, $UHU^\dagger = -(\mu B_0/2)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \equiv H'$
   b) $i\hbar U d\psi/dt = UHU^\dagger U\psi = H'\tilde{\psi}$,
   $U d\psi/dt = d\tilde{\psi}/dt - (dU/dt)\psi = d\tilde{\psi}/dt - (i\omega/2)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\tilde{\psi}$
   $\Rightarrow i\hbar d\tilde{\psi}/dt = H'\tilde{\psi} - (\hbar\omega/2)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\tilde{\psi}$
   c) $\tilde{\psi}(t) = \exp\left(-(i/\hbar)\tilde{H}t\right)\tilde{\psi}(0)$, with $\tilde{\psi}(0) = \psi(0) = (1,0)$.
   the exponent of the 2×2 matrix can be calculated using the given identity, with $a = \omega t/2$, $b = \mu B_0 t/2\hbar$, $r = (t/2\hbar)\sqrt{\hbar^2\omega^2 + \mu^2}$.
   The result is $u(t) = \cos r + i(a/r)\sin r$, $v(t) = i(b/r)\sin r$.

4. *(a)* free motion at constant velocity $\dot{x} = v = (x_2 - x_1)/(t_2 - t_1)$, so the action is $S_{\text{class}} = \frac{1}{2}mv^2(t_2 - t_1) = \frac{1}{2}m(x_2 - x_1)^2/(t_2 - t_1)$.

(b) insert a resolution of the identity $\int dp|p\rangle\langle p|$ to write

$$G = (2\pi\hbar)^{-1}\int_{-\infty}^{\infty} dp \exp\left(\frac{i}{\hbar}(x_2 - x_1)p\right)\exp\left(-\frac{i}{\hbar}(t_2 - t_1)\frac{p^2}{2m}\right)$$
$$= \exp\left(\frac{im(x_2 - x_1)^2}{2\hbar(t_2 - t_1)}\right)\sqrt{\frac{m}{2\pi i\hbar(t_2 - t_1)}}.$$

(c) Feynman's path integral formula: $G(x_2, t_2; x_1, t_1) \propto \sum_{\text{paths}} \exp\left(\frac{i}{\hbar}S_{\text{path}}\right)$. In the semiclassical limit only classical paths contribute; in this case there is a single classical path, so $G \propto e^{iS_{\text{class}}/\hbar}$; the proportionality constant contains fluctuations around the classical path.