ANSWERS TO THE EXAM QUANTUM THEORY, 3 JANUARY 2022

each item gives 2 points for a fully correct answer, grade = total $\times\, 9/24 + 1$

1. *a)* it is given that $\mathcal{T}^2|\psi\rangle = e^{i\phi}|\psi\rangle$; apply $\mathcal{T}$ to both sides of the equation, $\mathcal{T}^2\mathcal{T}|\psi\rangle = e^{-i\phi}\mathcal{T}|\psi\rangle$, but also $\mathcal{T}^2\mathcal{T}|\psi\rangle = e^{i\phi}\mathcal{T}|\psi\rangle$, so $e^{i\phi} = e^{-i\phi}$ and hence ϕ is either 0 or π.
   b) $U\psi = \lambda\psi$, define $\psi' = \mathcal{T}\psi$, then, because $\mathcal{T}$ commutes with U, we have $U\psi' = \mathcal{T}U\psi = \mathcal{T}\lambda\psi = \lambda^*\psi'$, so also λ^* is an eigenvalue of U if λ is an eigenvalue.

2. *a)* The wave function decays to zero for $x \to \pm\infty$, it is symmetric without a node for the ground state, it is symmetric with a cusp-like node at $x = 0$ for the first excited state, and symmetric with two nodes for the second excited state.
   b) There are two turning points where the velocity goes to zero smoothly, and these are associated with a phase shift of $-\pi/2$, so the total phase shift is $\gamma = -\pi$.
   c) $E = p_x^2/2m + V_0|x| \to p_x = \pm\sqrt{2m(E - V_0|x|)}$;
   $\oint p_x dx = 4\int_0^{E/V_0} \sqrt{2m(E - V_0 x)}dx = \frac{8}{3}(2m)^{1/2}V_0^{-1}E^{3/2} = 2\pi\hbar(n + 1/2)$;
   $E_n = (3\pi\hbar V_0/4)^{2/3}(2m)^{-1/3}(n + 1/2)^{2/3}$.

3. *a)* $d\vec{r}/dt = (i/\hbar)[H, \vec{r}] = (1/m)(\vec{p} - q\vec{A})$
   b) $U = \exp(iq\chi/\hbar)$, then $\exp(iq\chi/\hbar)(\vec{p}-q\vec{A})^2\exp(-iq\chi/\hbar) = [\exp(iq\chi/\hbar)(\vec{p}-q\vec{A})\exp(-iq\chi/\hbar)]^2 = (\vec{p} - q\vec{A} - q\nabla\chi)^2 = (\vec{p} - q\vec{A}')^2$
   c) The relation between H' and H is a unitary transformation, which leaves the eigenvalues unaffected. The eigenvectors ψ' of H' are related to those of H by $\psi' = U\psi$, such that if $H\psi = E\psi$, then $H'\psi' = UHU^\dagger U\psi = UH\psi = E\psi'$.

4. *a)* $b = a + \alpha$, $c = a - \alpha$; $[b, b^\dagger] = [c, c^\dagger] = [a, a^\dagger] = 1$.
   b) The operators b, c are bosonic annihilation operators, so the eigenvalues of $b^\dagger b$ and $c^\dagger c$ are those of the harmonic oscillator, the integers $0, 1, 2, \ldots$. These are then the eigenvalues of H, each twofold degenerate.
   c) $Q|\Psi_0\rangle = \begin{pmatrix} \alpha\sqrt{\beta} - \gamma\sqrt{\alpha} \\ \gamma\sqrt{\beta} - \beta\sqrt{\alpha} \end{pmatrix}|\gamma\rangle$, this must equal 0, hence $\gamma = \sqrt{\alpha\beta}$.
   d) The Hamiltonian $H = Q^\dagger Q$ is positive definite; any eigenvalue E must satisfy $E = \langle\Psi_0|H|\Psi_0\rangle = \langle\Psi'|\Psi'\rangle \geq 0$, with $|\Psi'\rangle = Q|\Psi\rangle$. Hence $E \geq 0$, so $E_0 = 0$ must be the lowest eigenvalue.