Answers to the Exam Quantum Theory, 8 January 2024

each item gives 2 points for a fully correct answer, grade = total $\times\, 9/24 + 1$

1. *a)* $a|\beta\rangle = C\sum_{n=0}^{\infty}\frac{\beta^n}{n!}a(a^\dagger)^n|0\rangle = C\sum_{n=1}^{\infty}\frac{n\beta^n}{n!}(a^\dagger)^{n-1}|0\rangle = \beta|\beta\rangle$.
   b) $\langle\beta|\beta\rangle = C^*\sum_{n=0}^{\infty}\frac{(\beta^*)^n}{n!}\langle 0|a^n|\beta\rangle = C^*\sum_{n=0}^{\infty}\frac{(\beta\beta^*)^n}{n!}\langle 0|\beta\rangle = |C|^2 e^{|\beta|^2} = 1$, so $C = e^{-|\beta|^2/2}$.
   c) Act on this state with $a^\dagger$, so raise the number of quanta in each term of the sum by 1 unit. Then the sum will no longer have the state $|0\rangle$ with zero quanta, so c_0 must be zero if $|\Psi\rangle$ is to be an eigenstate of $a^\dagger$. But if $c_0 = 0$, the state $|\Psi\rangle$ has no term with 0 quanta, hence $a^\dagger|\Psi\rangle$ has no term with 1 quantum, so also c_1 must be zero. And so on, all coefficients must be zero if $|\Psi\rangle$ is an eigenstate of $a^\dagger$, which does not give a valid eigenstate.

2. *a)* The energy separation between ground state and first excited state is $\delta E(t) = (3/2m)[\hbar\pi/L(t)]^2$ and we need $\hbar|d\delta E/dt| \ll (\delta E)^2$, so $dL/dt \ll \hbar/mL(t)$; the adiabatic approximation holds if $dL/dt \ll \hbar/mL_2$.
   b) The energy varies in time as $E(t) = (\hbar\pi/L(t))^2/2m$ from $t = 0$ to $t = T$, the dynamical phase is $(\hbar\pi^2/2m)\int_0^T L(t)^{-2}\,dt$.
   c) The wave function varies in time as $\psi(x,t) = (L(t)/2)^{-1/2}\sin(\pi x/L(t))$, Berry phase is $i\oint\langle\psi|d/dL|\psi\rangle dL$. We evaluate $\langle\psi|d/dL|\psi\rangle = \int_0^L dx\,\frac{1}{2}\frac{d}{dL}\psi(x,t)^2 = \frac{1}{2}\frac{d}{dL}\int_0^L dx\,\psi(x,t)^2 - \frac{1}{2}\psi(L,t)^2 = 0$.

3. *a)* Kramers theorem requires time-reversal symmetry, which means that H should be invariant if $\boldsymbol{p} \mapsto -\boldsymbol{p}$ and $\sigma_\alpha \mapsto -\sigma_\alpha$. A nonzero μ breaks that symmetry, so Kramers theorem does not apply.
   b) Squaring H produces a unit matrix times $v^2p_x^2 + v^2p_y^2 + \mu^2$, so the spectrum consists of the two branches $\pm\sqrt{v^2p_x^2 + v^2p_y^2 + \mu^2}$.
   c) Expansion near $\boldsymbol{p} = 0$ gives (for the positive energy branch) $\mu + v^2|\boldsymbol{p}|^2/2\mu$ plus higher order terms. This is the dispersion relation of a particle of mass μ/v^2.

4. *(a)* insert a resolution of the identity $\int dp|p\rangle\langle p|$ to write

$$G = (2\pi\hbar)^{-1}\int_{-\infty}^{\infty} dp\,\exp\left(\frac{i}{\hbar}(x_2 - x_1)p\right)\exp\left(-\frac{i}{\hbar}(t_2 - t_1)\frac{p^2}{2m}\right)$$
$$= \exp\left(\frac{im(x_2 - x_1)^2}{2\hbar(t_2 - t_1)}\right)\sqrt{\frac{m}{2\pi i\hbar(t_2 - t_1)}}.$$

   (a) free motion at constant velocity $\dot{x} = v = (x_2 - x_1)/(t_2 - t_1)$, so the action is $S_{\text{class}} = \frac{1}{2}mv^2(t_2 - t_1) = \frac{1}{2}m(x_2 - x_1)^2/(t_2 - t_1)$.
   (c) Feynman's path integral formula: $G(x_2, t_2; x_1, t_1) \propto \sum_{\text{paths}}\exp\left(\frac{i}{\hbar}S_{\text{path}}\right)$. In the semiclassical limit only classical paths contribute; in this case there is a single classical path, so $G \propto e^{iS_{\text{class}}/\hbar}$; the proportionality constant contains fluctuations around the classical path.