Answers to the Exam Quantum Theory, 29 January 2024

each item gives 2 points for a fully correct answer, grade = total $\times 9/24 + 1$

1. *a)* The time reversal operation inverts the sign of momentum and spin, so the Hamiltonian is unchanged.
   b) $E_\pm = \pm v p$. The velocity is dE/dp, so positive for E_+ and negative for E_-.
   c) Coupling of states from E_+ and E_- would open up a gap at $p = 0$, violating Kramers degeneracy.

2. *a)* $\bar{A} = \langle \Psi | A | \Psi \rangle = \langle e^{-iH_0 t/\hbar} \Psi_\mathrm{I} | e^{-iH_0 t/\hbar} A_\mathrm{I} e^{iH_0 t/\hbar} | e^{-iH_0 t/\hbar} \Psi_\mathrm{I} \rangle = \langle \Psi_\mathrm{I} | A_\mathrm{I} | \Psi_\mathrm{I} \rangle$.
   b) $dA_\mathrm{I}/dt = (i/\hbar) H_0 A_\mathrm{I} - (i/\hbar) A_\mathrm{I} H_0 = (i/\hbar)[H_0, A_\mathrm{I}]$.
   c) $i\hbar d\Psi/dt = H\Psi \rightarrow i\hbar d\Psi_\mathrm{I}/dt = -H_0 \Psi_\mathrm{I} + e^{iH_0 t/\hbar} H \Psi = (-H_0 + H_\mathrm{I})\Psi_\mathrm{I} = V_\mathrm{I} \Psi_\mathrm{I}$.

3. *(a)* $\langle 0 | \hat{x}^2 | 0 \rangle = (1/2) \langle 0 | \hat{a} \hat{a}^\dagger | 0 \rangle = 1/2$, $\langle 0 | \hat{p}^2 | 0 \rangle = (1/2) \langle 0 | \hat{a} \hat{a}^\dagger | 0 \rangle = 1/2$
   (b) define $|\psi\rangle = \hat{a}^\dagger |N\rangle$, then

   $$\hat{a}^\dagger \hat{a} |\psi\rangle = \hat{a}^\dagger (\hat{a}^\dagger \hat{a} + 1) |N\rangle = (N+1) |\psi\rangle$$

   so $|\psi\rangle = C|N+1\rangle$. The coefficient $C = (N+1)^{1/2}$ follows from $\langle \psi | \psi \rangle = \langle N | \hat{a}^\dagger \hat{a} + 1 | N \rangle = N + 1 = C^2$.
   (c) the expectation values of $\hat{x}$ and $\hat{p}$ vanish because $\langle N | \hat{a}^\dagger | N \rangle \propto \langle N | N+1 \rangle = 0$ and $\langle N | \hat{a} | N \rangle \propto \langle N+1 | N \rangle = 0$.
   the second moment follows from $\langle N | \hat{x}^2 | N \rangle = \frac{1}{2} \langle 0 | \hat{a}^\dagger \hat{a} + \hat{a} \hat{a}^\dagger | 0 \rangle = \frac{1}{2}(2N+1)$, and similarly for the momentum.

4. *(a)* The momentum $p = mv + eA$ in the Bohr-Sommerfeld rule is the canonical momentum, not just the mechanical momentum.
   (b) The contribution to $\oint p \cdot dq$ from the electromagnetic momentum is $e \oint A \cdot dq = -eB\pi l_\mathrm{cycl}^2 = -2\pi m E/eB$, which gives $E_n = \hbar\omega_n (n + 1/2)$. It differs from Alvaro's answer by a factor two.
   (c) For massless electrons we have $E = pv$, $l_\mathrm{cycl} = p/eB$, so $\oint p \cdot dq = p \times 2\pi l_\mathrm{cycl} - eB \times \pi l_\mathrm{cycl}^2 = \pi p^2/eB = \pi E^2/(eBv^2)$; the quantization is $E_n^2 = 2\hbar e B v^2 (n + \gamma)$; the offset $\gamma = 0$ because the phase shift from the turning points is canceled by the Berry phase.