Answers to the Exam Quantum Theory, 18 February 2026

each item gives 2 points for a fully correct answer, grade = total $\times\, 9/24 + 1$

1. *(a)* An operator O is unitary if $O^\dagger = O^{-1}$, hence if $\langle\psi|O\phi\rangle = \langle O^{-1}\psi|\phi\rangle$. In the case of translation we have $T_a^{-1} = T_{-a}$, hence we need

$$\int_{-\infty}^{\infty} \psi^*(x)\phi(x+a)\,dx = \int_{-\infty}^{\infty} \psi^*(x-a)\phi(x)\,dx,$$

which is indeed the case (change coordinates to $x' = x - a$).
(b) Eigenstates $\psi(x) = e^{ikx}$ have eigenvalue e^{ika}, for any real k.
(c) If $H\psi(x) = E\psi(x)$, and $[H, T_a] = 0$, then also $H\psi'(x) = E\psi'(x)$, with $\psi'(x) = T_a\psi(x) = \psi(x+a)$. The eigenstate is nondegenerate, so $\psi'(x) = \lambda\psi(x)$ for some $\lambda \in \mathbb{C}$, and this means that $\psi(x+a) = \lambda\psi(x)$.

2. *a)* $S^\dagger(s) = \exp\left(\frac{1}{2}sa^\dagger a^\dagger - \frac{1}{2}saa\right) = S(s)^{-1} \neq S(s)$, so this operator is unitary but not Hermitian.
b) $\langle s|\hat{p}|s\rangle = \langle 0|S^\dagger(s)\hat{p}S(s)|0\rangle$
$= 2^{-1/2}i\langle 0|(a^\dagger \cosh s - a\sinh s) - (a\cosh s - a^\dagger \sinh s)|0\rangle = 0$
$\langle s|\hat{p}^2|s\rangle = -(1/2)\langle 0|\left((a^\dagger \cosh s - a\sinh s) - (a\cosh s - a^\dagger\sinh s)\right)^2|0\rangle$
$= (1/2)(\cosh s + \sinh s)^2 = (1/2)e^{2s}$.
c) there is no contradiction: the uncertainty principle provides a lower bound to the product of the variance of position and momentum, so if the variance of momentum goes to zero for $s \to -\infty$, the variance of position must diverge as e^{-2s}.

3. *(a)* H is Hermitian because $(d/dx)^\dagger = -d/dx$, so $(id/dx)^\dagger = id/dx$ and $H = H^\dagger$. We may write $H = vp\sigma_z$, time reversal exchanges $p \mapsto -p$ and $\sigma_\alpha \mapsto -\sigma_\alpha$, so H is time reversally symmetric. Alternatively, show that H commutes with the time reversal operator $\sigma_y K$, with K the complex conjugation operator.
(b) The eigenvalue is $E_\pm = \pm\hbar v k$ for the two choices of the eigenstate. The velocity is $(1/\hbar)dE/dk$, so positive for E_+ and negative for E_-.
(c) For $H = vp\sigma_z + \mu\sigma_x$ one has $H^2 = (vp)^2 + \mu^2$ times the unit matrix, so the energy spectrum is $E = \pm\sqrt{(\hbar v k)^2 + \mu^2}$, with a gap 2μ around $E = 0$.

4. *a)* $d\vec{r}/dt = (i/\hbar)[H, \vec{r}] = (1/m)(\vec{p} - q\vec{A})$
b) $\exp(iq\chi/\hbar)(\vec{p}-q\vec{A})^2\exp(-iq\chi/\hbar) = [\exp(iq\chi/\hbar)(\vec{p}-q\vec{A})\exp(-iq\chi/\hbar)]^2 = (\vec{p} - q\vec{A} - q\nabla\chi)^2 = (\vec{p} - q\vec{A}')^2$
c) H' is related to H by a unitary transformation, so it represents the same system described in a different basis.