EXAM QUANTUM THEORY, 22 DECEMBER 2014, 10–13 HOURS.

1. Assume that the Hamiltonian $H = H_0 + V$ is the sum of two time-independent Hermitian operators H_0 (called the "free" part) and V (called the "interaction" part). In the socalled "interaction picture" the time-dependent state $\Psi(t)$ and operator A are transformed as

$$\Psi_{\rm I}(t) = e^{iH_0 t/\hbar}\Psi(t), \quad A_{\rm I}(t) = e^{iH_0 t/\hbar} A e^{-iH_0 t/\hbar}.$$

- *a)* Explain why the transformation to the interaction picture has no effect on the expectation value $\bar{A}(t) = \langle\Psi(t)|A|\Psi(t)\rangle$ of an operator A in the state $\Psi(t)$, so that we may equally well write $\bar{A}(t) = \langle\Psi_{\rm I}(t)|A_{\rm I}(t)|\Psi_{\rm I}(t)\rangle$.
- *b)* Derive the Heisenberg equation of motion in the interaction picture:

$$i\hbar\frac{d}{dt}A_{\rm I} = [A_{\rm I}, H_0].$$

- *c)* Show, starting from the Schrödinger equation for $\Psi(t)$, that the evolution equation for $\Psi_{\rm I}(t)$ can be written in the form

$$i\hbar\frac{d}{dt}\Psi_{\rm I}(t) = V_{\rm I}(t)\Psi_{\rm I}(t),$$

without any explicit dependence on H_0.

2. The socalled Rashba Hamiltonian describes the motion of an electron in a two-dimensional layer in the x-y plane, with a coupling of the spin to the motion. It acts on the spin-up and spin-down components $\Psi_\uparrow$, $\Psi_\downarrow$ of the wave function Ψ as a 2×2 matrix,

$$H = \begin{pmatrix} p_x^2/2m + p_y^2/2m + V(x,y) & \alpha p_x - i\alpha p_y \\ \alpha p_x + i\alpha p_y & p_x^2/2m + p_y^2/2m + V(x,y) \end{pmatrix}.$$

The momentum in the x-y plane has components $p_x = -i\hbar\partial/\partial x$, $p_y = -i\hbar\partial/\partial y$, the potential is $V(x,y)$, and the real coefficient α represents the spin-orbit coupling strength.

The time-reversed state $\tilde{\Psi}$ is defined from Ψ by a combination of complex conjugation and spin flip,

$$\begin{pmatrix} \tilde{\Psi}_\uparrow \\ \tilde{\Psi}_\downarrow \end{pmatrix} = \begin{pmatrix} \Psi_\downarrow^* \\ -\Psi_\uparrow^* \end{pmatrix}.$$

- *a)* Show that the time-reversal operation conserves the normalization of the state and show that $\Psi \mapsto -\Psi$ if you apply the time-reversal operation twice.

An eigenstate Ψ at energy E satisfies

$$H\begin{pmatrix} \Psi_\uparrow \\ \Psi_\downarrow \end{pmatrix} = E\begin{pmatrix} \Psi_\uparrow \\ \Psi_\downarrow \end{pmatrix}.$$

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- *b)* Show that the time reversed state $\tilde{\Psi}$ is an eigenstate of H at the same energy E.
- *c)* Prove that Ψ and $\tilde{\Psi}$ are linearly independent. (You have then proven the socalled Kramers degeneracy.)

3. The bosonic annihilation operator a and creation operator $a^\dagger$ can be used to describe the state of photons at a given frequency. For example, the coherent state is given by

$$|\beta\rangle = e^{-|\beta|^2/2} e^{\beta a^\dagger} |0\rangle,$$

where β is a complex number and $|0\rangle$ is the vacuum state. In what follows you may use that the coherent state is an eigenstate of the annihilation operator: $a|\beta\rangle = \beta|\beta\rangle$.

- *a)* Show that two coherent states $|\alpha\rangle$ and $|\beta\rangle$ are not orthogonal, by deriving that

$$|\langle\alpha|\beta\rangle|^2 = \exp(-|\alpha - \beta|^2).$$

- *b)* Calculate the first two moments of the photon number: $\bar{n} = \langle\beta|a^\dagger a|\beta\rangle$ and $\overline{n^2} = \langle\beta|(a^\dagger a)^2|\beta\rangle$, and the variance $\operatorname{var} n = \overline{n^2} - (\bar{n})^2$. Verify that the ratio of the variance and the average is equal to 1. (This ratio is called the Fano factor.)

The coherent state is a pure state. The photons can also be in a mixed state, described by a density matrix ρ. Consider, for example, a mixture of two coherent states,

$$\rho = p|\alpha\rangle\langle\alpha| + (1 - p)|\beta\rangle\langle\beta|,$$

with relative weight $p \in (0, 1)$.

- *c)* Calculate from this density matrix the first two moments of the photon number and show that the Fano factor $(\operatorname{var} n)/\bar{n}$ is greater than 1.

4. A particle moves along the x-axis in the potential $V(x) = V_0|x|$, with $V_0 > 0$.

- *a)* Make a sketch of the wave function $\Psi_n(x)$ for the ground state and the first two excited states. (Indicate which is which.) Pay particular attention to sign changes of $\Psi_n(x)$ and to the $\pm x$ symmetry.

We seek the energy spectrum in the Bohr-Sommerfeld approximation,

$$\frac{1}{\hbar}\oint p_x dx + \gamma = 2\pi n, \quad n = 0, 1, 2, \ldots$$

- *b)* What is the appropriate value of the phase shift γ?
- *c)* Calculate the energy levels E_n.