EXAM QUANTUM THEORY, RETAKE, 19 JANUARY 2015, 10–13 HOURS.

1. The density matrix ρ has the general expression

$$\rho = \sum_n p_n |\Psi_n\rangle\langle\Psi_n|.$$

The coefficients p_n are real and positive. Each state $|\Psi_n\rangle$ is normalized to unity, but pairs of states $|\Psi_n\rangle$ and $|\Psi_m\rangle$ need not be orthogonal.

- *a)* Derive that $\langle\psi|\hat{\rho}|\psi\rangle \geq 0$ for any arbitrary state $|\psi\rangle$.
- *b)* Show, using the Schrödinger equation with Hamiltonian H for $\Psi_n(t)$, that the density matrix evolves in time according to

$$i\hbar\frac{d}{dt}\rho(t) = H\rho(t) - \rho(t)H.$$

- *c)* The density matrix of a pure state satisfies $\rho^2 = \rho$. Show that a state is pure at time $t > 0$ if and only if it is pure at time $t = 0$.

2. A Hamiltonian $H(\lambda)$ depends on a parameter λ, so its eigenvalues $E_n(\lambda)$ and eigenfunctions $|n, \lambda\rangle$ are also λ-dependent.

- *a)* Prove the Hellmann-Feynman theorem relating the expectation value of the derivative of the Hamiltonian to the derivative of the energy,

$$\langle n, \lambda|\partial H/\partial\lambda|n, \lambda\rangle = \frac{d}{d\lambda}E_n(\lambda).$$

An electron moves in a wire of constant cross-section parallel to the z-axis, under the influence of a uniform magnetic field B in the x-direction. The Hamiltonian is given by

$$H = p_x^2/2m + p_y^2/2m + (p_z - eBy)^2/2m + V(x, y),$$

with a potential $V(x, y)$ that depends only on the coordinates perpendicular to the wire.

- *b)* Explain why the momentum p_z along the wire is a conserved quantity. How can you reconcile this with the fact that the electron is deflected by the magnetic field, so it does *not* move with a constant speed in the z-direction.
- *c)* If you are given the dependence $E(p_z)$ of the energy on p_z in an eigenstate of H, how can you determine the expectation value of the velocity along the wire?
Hint: the velocity operator in the z-direction is $v_z = (p_z - eBy)/m$.

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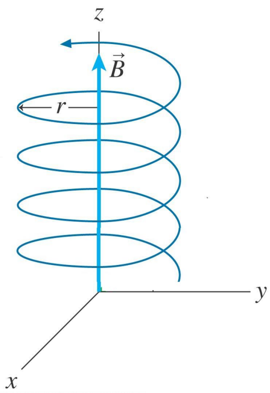
3. The classical two-dimensional motion of a particle of charge q, mass m in the x-y plane, under the influence of a perpendicular magnetic field B, is a circle of radius $R_c = mv/|qB|$, where v is the magnitude of the velocity of the particle along the circle. The semiclassical quantization of this circular motion says that the magnetic flux through the circle is quantized as $(n + \frac{1}{2})h/e$, with $n = 0, 1, 2, \ldots$.

- *a)* Derive the corresponding expression for the quantized energy E_n of the particle. Plot the ground state energy E_0 as a function of the magnetic field B (show both positive and negative values of B).
- *b)* In graphene the particles are massless Dirac fermions. Show in the same plot how this modifies the dependence of E_n on B and point out the two qualitative differences with respect to particles with $m \neq 0$.

Consider now the *three-dimensional* motion of a particle of charge q, mass $m \neq 0$, in a uniform magnetic field B pointing in the z-direction. The classical motion is a spiral around the z-axis. (See figure.) The energy spectrum $E_n(k)$ depends on a real variable k and on a discrete quantum number $n = 0, 1, 2, \ldots$.



- *c)* Sketch the first few functions $E_0(k)$, $E_1(k)$, and $E_2(k)$ in one single plot. How large is the spacing $E_{n+1}(0) - E_n(0)$ between subsequent levels?

4. The Hamiltonian $H(t)$ of an electron in a time-dependent magnetic field $\vec{B}(t)$ in the x-y plane is given in terms of Pauli matrices by

$$H(t) = -\frac{\hbar e}{2m}(B_x(t)\sigma_x + B_y(t)\sigma_y), \text{ with } \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

The magnetic field rotates with period T at constant magnitude B_0, according to $B_x(t) = B_0 \cos(2\pi t/T)$, $B_y(t) = B_0 \sin(2\pi t/T)$.

- *a)* Show that the two eigenvalues of $H(t)$ are $\pm\frac{1}{2}\hbar\omega_0$, with $\omega_0 = eB_0/m$.

We assume that the electron starts out at $t = 0$ in the state

$$\psi(0) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \equiv \psi_0$$

According to the adiabatic approximation, the state after one period T is given by $\psi(T) = e^{ic_0 + ic_1 T}\psi(0)$, where the coefficients c_0 and c_1 do not depend on T.

- *b)* Give the expression for the coefficient c_1. (No derivation required.)
- *c)* Calculate the coefficient c_0, starting from the formula for the Berry phase.

Hint 1: note that $\phi(t) = \sqrt{\frac{1}{2}}\begin{pmatrix} 1 \\ e^{2\pi i t/T} \end{pmatrix}$ is an eigenstate of $H(t)$ with $\phi(0) = \phi(T) = \psi_0$.

Hint 2: on a separate page you can find a summary of the lecture notes on the Berry phase, to refresh your memory.

this sheet is not part of the exam, but is for your information only

Lecture notes on the Berry phase

adiabatic evolution with return to starting point ("cyclic"): final state can only differ from initial state by a phase factor (for a nondegenerate state)

$$\psi_{\text{final}} = e^{i\gamma_{\text{B}}} \exp\left(-\frac{i}{\hbar}\int_0^T E(t)\,dt\right)\psi_{\text{initial}}$$

dynamical phase factor, depending on the period T of the cycle
Berry phase γ_{B} independent on T ("geometric" phase)

$$i\hbar\frac{d}{dt}|\psi(t)\rangle = H[\alpha(t)]\psi(t), \quad \text{eigenstate:} \quad H[\alpha(t)]|\alpha(t)\rangle = E(t)|\alpha(t)\rangle$$

$$\gamma_{\text{B}} = i\oint \left\langle \alpha \,\middle|\, \frac{\partial}{\partial\alpha} \,\middle|\, \alpha \right\rangle d\alpha$$