Exam Quantum Theory, 21 december 2015, 10–13 hours.

1. The states $|0\rangle$, $|1\rangle$, $|2\rangle$ denote orthonormal states of a quantum system, with density matrix $\hat{\rho}$.

- *a)* List three conditions that *any* valid density matrix should satisfy.
- *b)* Explain, for each of the matrices $\hat{\rho}_1$, $\hat{\rho}_2$, $\hat{\rho}_3$, if it is a valid density matrix or not:

$$\hat{\rho}_1 = \tfrac{1}{3}|0\rangle\langle 0| - \tfrac{1}{3}|1\rangle\langle 1| + |2\rangle\langle 2|, \quad \hat{\rho}_2 = |0\rangle\langle 0| + |0\rangle\langle 1|,$$

$$\hat{\rho}_3 = \tfrac{1}{2}|0\rangle\langle 0| - \tfrac{1}{2}|0\rangle\langle 1| - \tfrac{1}{2}|1\rangle\langle 0| + \tfrac{1}{2}|1\rangle\langle 1|.$$

- *c)* Explain, for each of the density matrices $\hat{\rho}_4$, $\hat{\rho}_5$, $\hat{\rho}_6$, if it represents a pure state or not:

$$\hat{\rho}_4 = \tfrac{1}{2}|0\rangle\langle 0| + \tfrac{1}{2}|0\rangle\langle 1| + \tfrac{1}{2}|1\rangle\langle 0| + \tfrac{1}{2}|1\rangle\langle 1|, \quad \hat{\rho}_5 = \tfrac{1}{2}|0\rangle\langle 0| + \tfrac{1}{2}|1\rangle\langle 1|,$$

$$\hat{\rho}_6 = \tfrac{1}{4}|0\rangle\langle 0| + \tfrac{1}{4}|1\rangle\langle 1| + \tfrac{1}{2}|2\rangle\langle 2|.$$

2. The operators $\hat{a}, \hat{a}^\dagger$ are bosonic annihilation and creation operators. The position operator $\hat{x}$ and momentum operator $\hat{p}$ are given by

$$\hat{x} = \sqrt{\tfrac{1}{2}}(\hat{a} + \hat{a}^\dagger), \quad \hat{p} = \sqrt{\tfrac{1}{2}}\, i(\hat{a}^\dagger - \hat{a}).$$

We define, for a given real number r, the socalled “squeeze operator”

$$\hat{S}(r) = \exp\left(\tfrac{1}{2} r \hat{a}^2 - \tfrac{1}{2} r (\hat{a}^\dagger)^2\right).$$

The socalled “squeezed vacuum” is defined by $|r\rangle = \hat{S}(r)|0\rangle$, where $|0\rangle$ is the vacuum state.

- *a)* Prove that the squeeze operator is a unitary operator.

In what follows you may use the following result for the unitary transformation of the annihilation operator:

$$\hat{S}^\dagger(r)\hat{a}\hat{S}(r) = \hat{a}\cosh r - \hat{a}^\dagger \sinh r.$$

- *b)* Calculate the first two moments $\overline{x} = \langle r|\hat{x}|r\rangle$ and $\overline{x^2} = \langle r|\hat{x}^2|r\rangle$ of the position operator, and show that the variance is given by

$$\operatorname{Var} x \equiv \overline{x^2} - (\overline{x})^2 = \tfrac{1}{2}e^{-2r}.$$

A similar calculation, which you do not need to perform yourself, gives $\operatorname{Var} p = \tfrac{1}{2}e^{2r}$.

- *c)* Discuss these results for $\operatorname{Var} x$ and $\operatorname{Var} p$ in connection with the Heisenberg uncertainty principle.

continued on second page

3. A particle of mass m moves along the x-axis in the potential well $V(x)$. At energy E the turning points of the motion are at $x = a$ and $x = b$ (with $a < b$). We analyze this quantum mechanical problem in the semiclassical Bohr-Sommerfeld approximation.

   The following integral may be useful in your calculations: $\int_0^1 \sqrt{1-x^2}\,dx = \pi/4$.

- *a)* Explain why the number of levels $N(E)$ with energy less than E is given approximately by

$$N(E) \approx \frac{\sqrt{2m}}{\pi\hbar}\int_a^b \sqrt{E - V(x)}\,dx.$$

  When is this a good approximation?

- *b)* Calculate the density of states $\rho(E) = dN/dE$ and relate it to the time T it takes the particle to move from one turning point to the other at energy E.

- *c)* The Bohr-Sommerfeld approximation can be improved by taking into account the phase shift γ acquired at the turning points. Use this improvement to calculate the lowest energy level in a parabolic potential well, given by $V(x) = cx^2$ (with coefficient $c > 0$).

4. A particle (charge q, mass m) in a magnetic field $\vec{B}(\vec{r}) = \nabla \times \vec{A}(\vec{r})$ has Hamiltonian

$$H = \frac{1}{2m}(\vec{p} - q\vec{A})^2, \quad \text{with} \quad \vec{p} = -i\hbar\nabla.$$

- *a)* Derive the Heisenberg equation of motion for the position operator $\vec{r}$, to obtain an expression for the velocity operator $\vec{v}$.

  We now investigate the effect of a gauge transformation of the vector potential, $\vec{A}'(\vec{r}) = \vec{A}(\vec{r}) + \nabla\chi(\vec{r})$, for some arbitrary function $\chi(\vec{r})$. The Hamiltonian with $\vec{A}$ replaced by $\vec{A}'$ is denoted by H'.

- *b)* Verify that H and H' are related by

$$H' = \exp(iq\chi/\hbar)H\exp(-iq\chi/\hbar).$$

- *c)* Explain why this relation between H and H' expresses the fact that the vector potentials $\vec{A}$ and $\vec{A}'$ describe the same physical system.