Exam Quantum Theory, 9 January 2017, 10–13 hours.

1. The harmonic oscillator (frequency ω) has Hamiltonian

$$\hat{H} = \tfrac{1}{2}\hbar\omega(\hat{x}^2 + \hat{p}^2),$$

in terms of the (dimensionless) position and momentum operators $\hat{x}$ and $\hat{p}$, with commutator $[\hat{x}, \hat{p}] = i$. In the Heisenberg representation each operator $\hat{O}$ becomes time dependent according to

$$\hat{O}(t) = e^{i\hat{H}t/\hbar}\hat{O}e^{-i\hat{H}t/\hbar}, \text{ with } \hat{O} \equiv \hat{O}(0).$$

- *a)* Consider an eigenstate $|n\rangle$ of $\hat{H}$ with energy E_n and another eigenstate $|n'\rangle$ with energy $E_{n'}$. Show that

$$\langle n'|\hat{x}(t)|n\rangle = \exp\left(\frac{it(E_{n'} - E_n)}{\hbar}\right)\langle n'|\hat{x}|n\rangle. \tag{1}$$

- *b)* Derive the Heisenberg equations of motion

$$\frac{d}{dt}\hat{x}(t) = \omega\hat{p}(t), \quad \frac{d}{dt}\hat{p}(t) = -\omega\hat{x}(t),$$

and derive the solution $\hat{x}(t) = \hat{x}\cos\omega t + \hat{p}\sin\omega t$.

- *c)* From this solution it follows that

$$\langle n'|\hat{x}(t)|n\rangle = \tfrac{1}{2}e^{i\omega t}Q_- + \tfrac{1}{2}e^{-i\omega t}Q_+, \text{ with } Q_\pm = \langle n'|\hat{x} \pm i\hat{p}|n\rangle. \tag{2}$$

Equate the two representations (1) and (2) of the matrix element to deduce that *if* $E_{n'} > E_n$, then *either* $\langle n'|\hat{x}|n\rangle = 0$ *or* $E_{n'} = E_n + \hbar\omega$.

2. The dimensionless position and momentum operators can be written in terms of the bosonic creation and annihilation operators $\hat{a}^\dagger$ and $\hat{a}$ (with commutator $[\hat{a}, \hat{a}^\dagger] = 1$), as follows:

$$\hat{x} = \sqrt{\tfrac{1}{2}}(\hat{a}^\dagger + \hat{a}), \quad \hat{p} = \sqrt{\tfrac{1}{2}}i(\hat{a}^\dagger - \hat{a}).$$

The vacuum state $|0\rangle$ is defined by $\hat{a}|0\rangle = 0$. Note that the expectation values of $\hat{x}$ and $\hat{p}$ vanish in the vacuum state.

- *a)* Derive the minimal uncertainty relation in the vacuum state,

$$\langle 0|\hat{x}^2|0\rangle\langle 0|\hat{p}^2|0\rangle = \tfrac{1}{4}.$$

The N-particle Fock state $|N\rangle$, normalized to $\langle N|N\rangle = 1$, is defined by $\hat{a}^\dagger\hat{a}|N\rangle = N|N\rangle$, with $N = 1, 2, 3, \ldots$.

- *b)* Derive, starting from this definition, the recursion relation

$$\hat{a}^\dagger|N\rangle = (N+1)^{1/2}|N+1\rangle.$$

- *c)* Explain why the recursion relation implies that the expectation values of $\hat{x}$ and $\hat{p}$ are zero in a Fock state. Then show that a Fock state is not a state of minimal uncertainty, by deriving that

$$\langle N|\hat{x}^2|N\rangle\langle N|\hat{p}^2|N\rangle = \tfrac{1}{4}(2N+1)^2.$$

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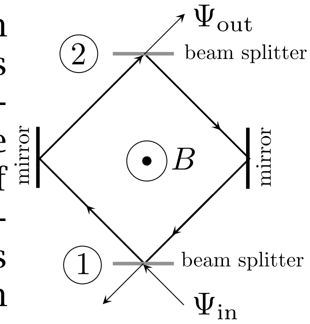
3. We consider the ring-shaped conductor shown in the figure. Electrons can enter into the ring, with wave amplitude Ψ_{in}, at beam splitter 1 and they can exit the ring, with amplitude Ψ_{out}, at beam splitter 2. The beam splitters have equal probability 1/2 for transmission and reflection. The electrons go around the ring in a clockwise direction, at wave number k and energy E. Each arm of the square-shaped ring has length L, so the ring has circumference $4L$ and area L^2. A magnetic field B points perpendicular to the ring. We study the transmission probability $T = |\Psi_{\text{out}}|^2/|\Psi_{\text{in}}|^2$.



- *a)* Consider first the case of zero magnetic field. Assume that an electron going around the ring once accumulates a phase shift ϕ. Explain how to arrive at this semiclassical formula for the transmission probability:

$$T = \frac{1}{5 - 4\cos\phi}.$$

- *b)* For $\phi = 0$ the transmission probability through the two beam splitters equals 1, even though each beam splitter separately only transmits with probability 1/2. How can one understand this? Explain also why for $\phi = \pi$ the transmission probability is much smaller than $1/2 \times 1/2 = 1/4$.
- *c)* Sketch how the transmission probability depends on the magnetic field. *Try to be specific:* For example, if the dependence is a monotonic decay, indicate the values of the low-field and high-field asymptotes. Or if the dependence is oscillatory, give the amplitude and period of the oscillation.

4. A particle of mass m moves freely along the x-axis, with Hamiltonian $H(x,p) = \frac{1}{2}p^2/m$ and Lagrangian $L(x,\dot{x}) = \frac{1}{2}m\dot{x}^2$.

- *a)* Calculate the classical action $S_{\text{class}} = \int_{t_1}^{t_2} L dt$ for the classical path from point x_1 at time t_1 to point x_2 at time t_2.
- *b)* Calculate the quantum mechanical propagator[1]

$$G(x_2, t_2; x_1, t_1) = \langle x_2 | e^{-(i/\hbar)(t_2 - t_1)\hat{H}} | x_1 \rangle.$$

- *c)* Discuss the relation between G and S_{class} in the context of Feynman's path integral formula.

[1]You may use the integral $\int_{-\infty}^{\infty} e^{ias - ibs^2} ds = \sqrt{\frac{\pi}{ib}} \exp\left(\frac{ia^2}{4b}\right)$.