Exam Quantum Theory, 13 February 2017, 14–17 hours.

1. The density matrix $\hat{\rho}$ of a system with Hamiltonian $\hat{H}$ evolves in time according to

$$i\hbar\frac{\partial}{\partial t}\hat{\rho}(t) = [\hat{H}, \hat{\rho}(t)],$$

where $[\cdots]$ denotes the commutator.

- *a)* Given that $\mathrm{Tr}\,\rho(t) = 1$ at $t = 0$, prove that this normalization condition holds for all $t > 0$.
- *b)* Given that $\hat{\rho}^2(t) = \hat{\rho}(t)$ at $t = 0$, prove that this purity condition holds for all $t > 0$.
- *c)* A state with $\hat{\rho}^2 = \hat{\rho}$ can be described by a certain wave function ψ. How are $\hat{\rho}$ and ψ related? Prove that $\hat{\rho}\psi = \psi$.

2. The parity operator $\hat{P}$ can be defined by its action on a wave function $\psi(x)$: $\hat{P}\psi(x) = \psi(-x)$.

- *a)* Recall the definition of a Hermitian operator and prove that $\hat{P}$ is Hermitian.
- *b)* Show that $\hat{P}$ is also unitary and give its eigenvalues.
- *c)* The Hamiltonian $\hat{H} = \hat{p}^2/2m + V(\hat{x})$ commutes with $\hat{P}$ if the potential $V(x)$ is an even function of x. Assume that this is the case and prove that the wave function of any nondegenerate energy level must be either an even or an odd function of x. (In your proof, indicate explicitly where you use the nondegeneracy of the energy level.)

3. A particle of charge e has Hamiltonian

$$H = \frac{1}{2m}\left(p - eA(q)\right)^2,$$

where for ease of notation we omit the *hat* ˆ on the operators. The particle moves along a line with coordinate q and momentum $p = -i\hbar d/dq$, in the presence of a vector potential $A(q)$. The substitution of A by $\tilde{A} = A + df/dq$, for some arbitrary function $f(q)$, is a gauge transformation.

- *a)* Show that the transformed Hamiltonian $\tilde{H}$ is related to H by a unitary transformation, $\tilde{H} = U^{-1}HU$.
- *b)* For $A = 0$ the lowest energy of the particle is $E_0 = 0$. Now take $A(q) = A_0 q$ and calculate E_0 as a function of A_0.

continued on second page

Instead of particle moving along a line, we next consider a particle moving along a circle in the x-y plane. If the radius R is large enough, we can use the same Hamiltonian H as before, with q now measuring the distance along the perimeter of the circle (so q advances by $2\pi R$ when the particle goes once around the circle). The vector potential is $A = BR/2$ with B the magnetic field in the z-direction.

- *c)* Plot the lowest energy E_0 of the particle as a function of B. Explain why you cannot make a gauge transformation to $\tilde{A} = 0$ and conclude that E_0 is B-independent.

4. The Hamiltonian of electrons in graphene is a 2×2 matrix,

$$\hat{H} = \begin{pmatrix} 0 & v(\hat{p}_x - i\hat{p}_y) \\ v(\hat{p}_x + i\hat{p}_y) & 0 \end{pmatrix} \tag{1}$$

where $v > 0$ is a constant velocity and $\hat{p}_x, \hat{p}_y$ are the two components of the momentum operator in the x-y plane. (There is no motion in the z-direction.)

- *a)* Calculate the energy spectrum $E(p_x, p_y)$ of graphene. Is there a lowest energy? *Hint: First calculate $\hat{H}^2$.*

In the presence of a uniform magnetic field B in the z-direction, the Hamiltonian of graphene is modified by the substitution $p_y \mapsto p_y - eBx$. The energy spectrum now consists of Landau levels.

- *b)* Show that there exists a B-independent Landau level at energy $E = 0$. *Hint: See if you can construct a zero-energy wave function of either the form*

$$\psi_1(x,y) = \begin{pmatrix} 0 \\ e^{iky} f(x) \end{pmatrix} \text{ or of the form } \psi_2(x,y) = \begin{pmatrix} e^{iky} f(x) \\ 0 \end{pmatrix},$$

for some constant k and some function $f(x)$.

- *c)* The classical motion of an electron in a magnetic field is a cyclotron orbit and the Landau level then follows from the quantization of this periodic motion. Explain the existence of an $E = 0$ Landau level in graphene from this semiclassical point of view.