EXAM QUANTUM THEORY, 5 JANUARY 2018, 14–17 HOURS.

1. The z-component S_z of the angular momentum operator of a spin-1 particle has three eigenvalues, $+1$, 0, -1, with eigenstates $|+1\rangle$, $|0\rangle$, $|-1\rangle$. In this basis the operator S_z and the density matrix ρ of the particle are given by

$$S_z = \begin{pmatrix} +1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \rho = \frac{1}{4}\begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

- *a)* Is the particle in a pure state or in a mixed state? Motivate your answer.
- *b)* Calculate the expectation value of S_z.
- *c)* A measurement of S_z gives the value 0. What is the density matrix of the particle after the measurement? Is the state pure or mixed?

2. The aim of this problem is to calculate the energy spectrum of an electron (charge e, mass m, momentum p) moving in the x–y plane in the presence of a magnetic field B in the z-direction. The Hamiltonian is

$$H = \frac{1}{2m}p_x^2 + \frac{1}{2m}(p_y - eBx)^2.$$

- *a)* Explain why the different Hamiltonian

$$H' = \frac{1}{2m}(p_x + \tfrac{1}{2}eBy)^2 + \frac{1}{2m}(p_y - \tfrac{1}{2}eBx)^2$$

has the same energy spectrum as H.

- *b)* Derive that the operator

$$a = (2e\hbar B)^{-1/2}(p_x + ip_y - ieBx)$$

satisfies the commutation relation $[a, a^\dagger] = 1$.

- *c)* Show that the Hamiltonian H can be written as follows in terms of the operators a and $a^\dagger$:

$$H = \frac{\hbar eB}{m}(a^\dagger a + \tfrac{1}{2}).$$

Use your knowledge of the harmonic oscillator to conclude what are the energy eigenvalues E_n of H.

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3. The Bohr-Sommerfeld quantization condition reads

$$\frac{1}{\hbar}\oint p\cdot dq + \gamma = 2\pi n, \quad n = 0, 1, 2, \ldots.$$

We would like to apply this to the periodic cyclotron motion of an electron (charge e, mass m) in a plane perpendicular to a magnetic field B.
Nicandro thinks he knows the answer: The cyclotron orbit is a circle of radius $l_{\rm cycl} = mv/eB$, the kinetic energy is $E = \frac{1}{2}mv^2$, so $\oint p\cdot dq = mv \times 2\pi l_{\rm cycl} = 4\pi mE/eB$, which gives the quantization $E_n = \frac{1}{2}\hbar\omega_c(n - \gamma/2\pi)$, with $\gamma = -\pi$ from two turning points.
This is the wrong answer.

- *a)* Which error has Nicandro made?
- *b)* Give the correct calculation.
- *c)* How would the quantization differ if the electrons are massless, as they are in graphene?

4. Given a time-independent Hamiltonian H, one can construct the time-dependent operator

$$U(t) = \exp\left(-\frac{i}{\hbar}tH\right).$$

- *a)* Show that $U(t)$ is a unitary operator and check that the wave function $\psi(t) = U(t)\psi(0)$ satisfies the Schrödinger equation.
- *b)* The Hamiltonian $H = p^2/2m$ of a free particle is independent of the position x. (We assume motion along the x-axis.) Derive the following integral expression for the propagator $G(x, x_0; t) = \langle x|U(t)|x_0\rangle$ from x_0 to x in a time t:

$$G(x, x_0; t) = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} dp\, e^{(ip/\hbar)(x-x_0)-(it/\hbar)p^2/2m}.$$

Evaluate the $t \to 0$ limit of $G(x, x_0; t)$.

- *c)* The integral over p can be carried out and the result is given:

$$G(x, x_0; t) = (2\pi i\hbar t/m)^{-1/2}\exp\left(\frac{im}{2\hbar t}(x - x_0)^2\right).$$

Discuss how this result relates to Feynman's formula for the quantum mechanical propagator as a sum over paths weighted by the exponent $e^{iS/\hbar}$ of the action. (Distinguish classical from nonclassical paths in your discussion.)