Exam Quantum Theory, 16 January 2019, 14–17 hours.

1. Consider two Hermitian operators A and B with commutator

$$[A, B] = i\hbar,$$

and consider a state ψ such that $\langle\psi|A|\psi\rangle = 0$ and $\langle\psi|B|\psi\rangle = 0$. The "square uncertainties' are defined by $\Delta A^2 = \langle\psi|A^2|\psi\rangle$ and $\Delta B^2 = \langle\psi|B^2|\psi\rangle$. The Heisenberg uncertainty relation says that

$$\Delta A^2 \Delta B^2 \geq \hbar^2/4.$$

- *a)* Prove this relation by calculating $\langle\Phi|\Phi\rangle$ with

$$\Phi = \left(A + \frac{i\hbar}{2\Delta B^2} B\right)\psi.$$

In the early development of quantum mechanics, one tried to derive an "energy-time" uncertainty relation by attempting to construct a "time operator" τ satisfying $[H, \tau] = i\hbar$. As Pauli demonstrated, this attempt fails because the energy spectrum of H would then become unbounded, so there cannot be a ground state. In questions b) and c) you are asked to develop this argument, in two steps:

- *b)* First show that $[H, \tau] = i\hbar$ implies $[H, e^{i\omega\tau}] = -\hbar\omega e^{i\omega\tau}$ for any real constant ω.
- *c)* Let ψ be an eigenstate of H with eigenvalue E. Derive that $\psi' = e^{i\omega\tau}\psi$ is also an eigenstate of H, with eigenvalue $E - \hbar\omega$. Verify that ψ' is still normalizable. Hence conclude there cannot be a ground state.

2. A bound state in a superconductor is described by the Hamiltonian

$$H = -\mu(a_\uparrow^\dagger a_\uparrow + a_\downarrow^\dagger a_\downarrow) + \Delta a_\uparrow^\dagger a_\downarrow^\dagger + \Delta^* a_\downarrow a_\uparrow.$$

Here $a_\uparrow, a_\downarrow$ is the annihilation operator of an electron with spin up or down, and $a_\uparrow^\dagger, a_\downarrow^\dagger$ is the corresponding creation operator. The coefficient μ (the Fermi energy) is real, the coefficient Δ (the pair potential) is complex.

- *a)* The number operator is $N = a_\uparrow^\dagger a_\uparrow + a_\downarrow^\dagger a_\downarrow$. Does N commute with H? Explain what your answer implies for particle number conservation.

The Hilbert space is spanned by the vacuum state $|\psi_1\rangle = |0\rangle$ and three more states $|\psi_2\rangle = a_\uparrow^\dagger a_\downarrow^\dagger|0\rangle$, $|\psi_3\rangle = a_\downarrow^\dagger|0\rangle$, $|\psi_4\rangle = a_\uparrow^\dagger|0\rangle$. The matrix h with elements $h_{nm} = \langle\psi_n|H|\psi_m\rangle$ has the form

$$h = \begin{pmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & a' & b' \\ 0 & 0 & c' & d' \end{pmatrix}.$$

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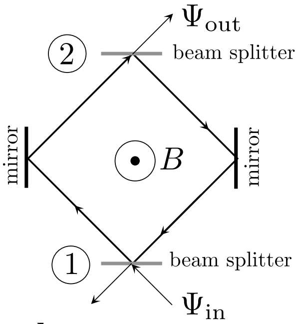
- *b)* Explain why the off-diagonal blocks of h are zero. Calculate the matrix elements $a, b, c, d, a', b', c', d'$ in the diagonal blocks. Check that the matrix you find is Hermitian.
- *c)* What are the four eigenvalues of H without superconductivity, for $\Delta = 0$? How does this spectrum change for nonzero Δ? Sketch the energy levels.

3. We consider the ring-shaped conductor shown in the figure. Electrons can enter into the ring (an $L \times L$ square) with wave amplitude Ψ_{in} at beam splitter 1 and they can exit the ring with amplitude Ψ_{out} at beam splitter 2. Each beam splitter transmits with probability T and reflects with probability $R = 1 - T$. The electrons go around the ring in a clockwise direction, and in one revolution they pick up a phase ϕ. We study the transmission probability $P = |\Psi_{\text{out}}|^2/|\Psi_{\text{in}}|^2$.



- *a)* Explain how to arrive at this semiclassical formula for the transmission probability:

$$P = \frac{T^2}{1 + R^2 - 2R\cos\phi}.$$

- *b)* For $\phi = 0$ the transmission probability through the two beam splitters equals 1, even if each beam splitter separately only transmits with probability $T \ll 1$. How can one understand this?
- *c)* Sketch how the transmission probability depends on a magnetic field B, perpendicular to the ring. *Try to be specific:* For example, if the dependence is a monotonic decay, indicate the values of the low-field and high-field asymptotes. Or if the dependence is oscillatory, give the amplitude and period of the oscillation.

4. A particle moves along the x-axis in the potential $V(x) = V_0 x^2$, with $V_0 > 0$.

- *a)* Make a sketch of the wave function $\Psi_n(x)$ for the ground state and the first two excited states. (Indicate which is which.) Pay particular attention to sign changes of $\Psi_n(x)$ and to the $\pm x$ symmetry.

  We seek the energy spectrum in the Bohr-Sommerfeld approximation,

$$\frac{1}{\hbar}\oint p_x dx + \gamma = 2\pi n, \quad n = 0, 1, 2, \ldots$$

- *b)* What is the appropriate value of the phase shift γ?
- *c)* Calculate the energy levels E_n.[1]

[1] You may want to use the integral $\int_0^1 \sqrt{1 - x^2}dx = \pi/4$.