EXAM QUANTUM THEORY, 12 MARCH 2020, 10–13 HOURS.

1. The operators of position $\hat{x}$ and of momentum $\hat{p}$ act as follows on a wave function $\psi(x)$ in the position representation:

$$\hat{x}\psi(x) = x\psi(x),\ \hat{p}\psi(x) = -i\hbar\tfrac{d}{dx}\psi(x).$$

- *a)* How do $\hat{x}$ and $\hat{p}$ act on a wave function $\Phi(p)$ in the momentum representation?
- *b)* Show that the commutator $[\hat{x}, \hat{p}]$ is the same in position and momentum representation.
- *c)* Are these two operators

$$\hat{O}_1 = \hat{x}\hat{p},\ \ \hat{O}_2 = \hat{x}^2\hat{p} - i\hbar\hat{x}$$

Hermitian or not? Motivate your answer.

2. Consider a Hamiltonian $\hat{H}$ with eigenvalues $E_0 \leq E_1 \leq E_2 \cdots$. The variational principle which we discussed in class says that the ground state energy E_0 has the *upper bound* $U = \langle\phi|\hat{H}|\phi\rangle$, where $|\phi\rangle$ is an arbitrary "trial" wave function, normalized by $\langle\phi|\phi\rangle = 1$.

- *a)* Expand the trial wave function $|\phi\rangle = \sum_{n=0}^{\infty} c_n|\psi_n\rangle$ in terms of the eigenfunctions $|\psi_n\rangle$ at energy E_n of $\hat{H}$ and derive that $U = \sum_{n=0}^{\infty}|c_n|^2 E_n$. Why does this imply that $E_0 \leq U$?

We will now derive a different principle, which says that $\hat{H}$ has at least one eigenvalue E_p in the interval

$$U - \sqrt{V - U^2} \leq E_p \leq U + \sqrt{V - U^2}, \tag{1}$$

where $V = \langle\phi|H^2|\phi\rangle = \sum_{n=0}^{\infty}|c_n|^2 E_n^2$.

- *b)* Prove that

$$V - U^2 = \sum_{n=0}^{\infty}|c_n|^2(E_n - U)^2.$$

- *c)* Denote by E_p the eigenvalue of $\hat{H}$ that is closest to U. Derive that

$$V - U^2 \geq (E_p - U)^2$$

and show that this implies the two inequalities in equation (1). Why is it wrong to conclude that the ground state energy has the *lower bound* $U - \sqrt{V - U^2}$?

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3. The bosonic annihilation operator $\hat{a}$ and creation operator $\hat{a}^\dagger$ can be used to describe the state of photons at a given frequency. A laser emits photons in a socalled coherent state, given by

$$|\beta\rangle = e^{-|\beta|^2/2} e^{\beta \hat{a}^\dagger} |0\rangle.$$

Here β is a complex number and $|0\rangle$ is the vacuum state.

- *a)* Show that the coherent state is an eigenstate of the annihilation operator: $\hat{a}|\beta\rangle = \beta|\beta\rangle$.
- *b)* Show that two coherent states $|\alpha\rangle$ and $|\beta\rangle$ are not orthogonal, by deriving that

$$|\langle\alpha|\beta\rangle|^2 = \exp(-|\alpha - \beta|^2).$$

- *c)* Calculate the first two moments of the photon number: $\bar{n} = \langle\beta|\hat{a}^\dagger\hat{a}|\beta\rangle$ and $\overline{n^2} = \langle\beta|(\hat{a}^\dagger\hat{a})^2|\beta\rangle$, and the variance $\operatorname{var} n = \overline{n^2} - (\bar{n})^2$. Discuss how your result agrees with the expected Poisson statistics of photons in a coherent state.

4. We consider the Hamiltonian $\hat{H} = \hat{p}^2/2m + V(\hat{x})$ of a particle of mass m moving along the x-axis in a confining potential $V(x)$. The eigenvalues of $\hat{H}$ form the discrete spectrum $E_0, E_1, E_2, \ldots$. We define the density of states $\rho(E) = \sum_{n=0}^{\infty} \delta(E - E_n)$ and its Fourier transform

$$F(t) = \int_{-\infty}^{\infty} \rho(E) e^{-iEt/\hbar} dE = \sum_{n=0}^{\infty} e^{-iE_n t/\hbar}.$$

The dynamics from position x_0 to x_1 in a time t is described by the propagator

$$G(x_1, x_0; t) = \langle x_1 | e^{-i\hat{H}t/\hbar} | x_0\rangle.$$

- *a)* Derive the following relation between $F(t)$ and the integral of the propagator for equal initial and final position:

$$\int_{-\infty}^{\infty} G(x, x; t) dx = F(t).$$

Feynman showed that the propagator $G(x_1, x_0; t)$ can be written as an integral over all paths $x(t')$ with $x(0) = x_0$ and $x(t) = x_1$,

$$G(x_1, x_0; t) = \sqrt{\frac{m}{2\pi i \hbar t}} \int_{x(0)=x_0}^{x(t)=x_1} \mathcal{D}[x(t')]\, e^{iS[x(t')]/\hbar}.$$

- *b)* What is the definition of $S[x(t')]$, how does the potential $V(x)$ enter in this definition?
- *c)* Suppose that $V(x)$ is a square well potential, so $V(x) = 0$ for $0 < x < W$, while $V(x) \to \infty$ for $x < 0$ and $x > W$. Draw in an x–t diagram a path $x(t)$ that contributes predominantly to the density of states $\rho(E)$ in the limit $\hbar \to 0$.