EXAM QUANTUM THEORY, 3 JANUARY 2022, 14.15–17.15 HOURS.

1. .

- *a)* You are given an *anti-unitary* operator $\mathcal{T}$ which satisfies $\mathcal{T}^2 = e^{i\phi}I$, with ϕ a real number and I the identity operator. Prove that $\mathcal{T}^2$ equals either $+I$ or $-I$.
- *b)* Consider a *unitary* operator U which commutes with $\mathcal{T}$. Assume that $\mathcal{T}^2 = +I$. Prove that if λ is an eigenvalue of U, then also the complex conjugate λ^* is an eigenvalue.

2. A particle moves along the x-axis in the potential $V(x) = V_0|x|$, with $V_0 > 0$.

- *a)* Make a sketch of the absolute value of the wave function $\Psi_n(x)$ for the ground state and the first two excited states. (Indicate which is which.) Pay particular attention to sign changes of $\Psi_n(x)$ and to the $\pm x$ symmetry.

  We seek the energy spectrum in the Bohr-Sommerfeld approximation,

$$\frac{1}{\hbar}\oint p_x dx + \gamma = 2\pi n, \quad n = 0, 1, 2, \ldots$$

- *b)* What is the appropriate value of the phase shift γ?
- *c)* Calculate the energy levels E_n.

3. A particle (charge q, mass m) in a magnetic field $\vec{B}(\vec{r}) = \nabla \times \vec{A}(\vec{r})$ has Hamiltonian

$$H = \frac{1}{2m}(\vec{p} - q\vec{A})^2, \text{ with } \vec{p} = -i\hbar\nabla.$$

- *a)* Determine the Heisenberg equation of motion for the position operator $\vec{r}$, to obtain an expression for the velocity operator $\vec{v}$.

  We now investigate the effect of a gauge transformation of the vector potential, $\vec{A}'(\vec{r}) = \vec{A}(\vec{r}) + \nabla\chi(\vec{r})$, for a given function $\chi(\vec{r})$. The Hamiltonian with $\vec{A}$ replaced by $\vec{A}'$ is denoted by H'.

- *b)* Verify that H and H' are related by $H' = UHU^\dagger$ for a certain unitary operator U.
- *c)* How are the eigenvalues of H and H' related? And how are the eigenfunctions related?

continued on second page

4. The Hamiltonian

$$H = \begin{pmatrix} \alpha^2 + a^\dagger a & \alpha a + \beta a^\dagger \\ \alpha a^\dagger + \beta a & \beta^2 + a^\dagger a \end{pmatrix} = Q^\dagger Q, \quad \text{for} \quad Q = \begin{pmatrix} \alpha & a \\ a & \beta \end{pmatrix}, \tag{1}$$

is known in quantum optics as a *Rabi Hamiltonian*. The operators $a^\dagger$ and a are bosonic creation and annihilation operators, the coefficients α, β are real numbers. The operator H is a 2×2 matrix which acts on the two-component wave function $\Psi = (\psi_1, \psi_2)$.

- *a)* Consider first the case $\alpha = \beta$. Show that the unitary transformation $H' = UHU^\dagger$ with $U = 2^{-1/2}\left(\begin{smallmatrix} 1 & 1 \\ -1 & 1 \end{smallmatrix}\right)$ brings the Hamiltonian to the diagonal form

$$H' = \begin{pmatrix} b^\dagger b & 0 \\ 0 & c^\dagger c \end{pmatrix}. \tag{2}$$

  How are the operators b and c related to a? Compute the commutators $[b, b^\dagger]$ and $[c, c^\dagger]$.
- *b)* Compute the eigenvalues of H for the case $\alpha = \beta$.
- *c)* Now consider the case of arbitrary real numbers α, β. Denote by $|\gamma\rangle$ the coherent state, such that $a|\gamma\rangle = \gamma|\gamma\rangle$, with γ an arbitary complex number. Find the value of γ such that the state $|\Psi_0\rangle = \begin{pmatrix} \sqrt{\beta}|\gamma\rangle \\ -\sqrt{\alpha}|\gamma\rangle \end{pmatrix}$ is an eigenstate of H with eigenvalue $E_0 = 0$.
- *d)* Prove that $|\Psi_0\rangle$ is the *ground state* of H, meaning that $E_0 = 0$ is the lowest eigenvalue.