EXAM QUANTUM THEORY, 24 JANUARY 2022, 14.15–17.15 HOURS.

1. In class we derived Kramers theorem for a Hermitian operator. Here we will derive a similar theorem for a unitary operator U. Let $\mathcal{T}$ be an antiunitary operator that squares to -1. Assume that

$$\mathcal{T}U = U^{-1}\mathcal{T}.$$

- *a)* Let Ψ be an eigenstate of U with eigenvalue λ. Prove that $\Psi' = \mathcal{T}\Psi$ is also an eigenstate of U with the same eigenvalue.
- *b)* Prove that Ψ and Ψ' are *not* linearly related, so that indeed the eigenvalue λ is doubly degenerate.

2. In class we encountered coherent states, which are eigenstates of the (bosonic) annihilation operator a. Alice and Bob wonder about eigenstates of the creation operator $a^\dagger$. Bob says: If $|\Psi\rangle$ is an eigenstate of a with eigenvalue λ, then $|\Psi\rangle^*$ is an eigenstate of $a^\dagger$ with eigenvalue λ^*. Alice disagrees.

- *a)* Who is right, Alice or Bob? Motivate your answer.
- *b)* Suppose that $|\beta\rangle$ is an eigenstate of $a^\dagger$ with eigenvalue $\beta \neq 0$. Prove that $\langle n+1|\beta\rangle = 0$ if $\langle n|\beta\rangle = 0$, for any number state $|n\rangle$.
- *c)* Prove that $\langle 0|\beta\rangle = 0$. What do you conclude about the existence of eigenstates of the creation operator? What about the special case $\beta = 0$?

3. We study a harmonic oscillator (frequency ω) with a time dependent mass* $m(t) = m_0 e^{\nu t}$, for some real constants m_0 and ν. Its Hamiltonian is

$$H(t) = \frac{p^2}{2m(t)} + \tfrac{1}{2}m(t)\omega^2 x^2.$$

(We work in the Schrödinger picture, so the operators x and p are not time dependent.)

- *a)* Define the operators $A = \frac{1}{2}\nu p x + \frac{1}{4}i\hbar\nu$ and $U(t) = e^{iAt/\hbar}$. Is A Hermitian? Is $U(t)$ unitary?
- *b)* Define $P(t) = U(t)pU^{-1}(t)$, $X(t) = U(t)xU^{-1}(t)$. Derive that

$$\frac{d}{dt}P(t) = -\frac{\nu}{2}P(t), \quad \frac{d}{dt}X(t) = \frac{\nu}{2}X(t).$$

Hint: notice that $[A,p] = \frac{1}{2}i\hbar\nu p$ *and* $[A,x] = -\frac{1}{2}i\hbar\nu x$.

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*This is known as the Caldirola-Kanai oscillator, and is used as a model for damping by friction.

- *c)* Solve these differential equations to obtain $P(t)$ and $X(t)$ in terms of p and x. Then prove that

$$H(t) = U(t)H(0)U^{-1}(t).$$

- *d)* Charlie looks at this expression for $H(t)$ and concludes that the wave function $\psi(t)$ of the harmonic oscillator depends on time as

$$\psi(t) = U(t)\exp[-(it/\hbar)H(0)]\psi(0).$$

This is not quite correct, what term has Charlie overlooked? Explain why Charlie's equation can be called the "adiabatic approximation" for $\psi(t)$.

4. A particle of mass m moves freely along the x-axis, with Hamiltonian $H(x,p) = \frac{1}{2}p^2/m$ and Lagrangian $L(x,\dot{x}) = \frac{1}{2}m\dot{x}^2$.

- *a)* Calculate the classical action $S_{\text{class}} = \int_{t_1}^{t_2} L dt$ for the classical path from point x_1 at time t_1 to point x_2 at time t_2.
- *b)* Calculate the quantum mechanical propagator†

$$G(x_2,t_2;x_1,t_1) = \langle x_2|e^{-(i/\hbar)(t_2-t_1)\hat{H}}|x_1\rangle.$$

- *c)* Discuss the relation between G and S_{class} in the context of Feynman's path integral formula.

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†You may use the integral $\int_{-\infty}^{\infty} e^{ias-ibs^2} ds = \sqrt{\frac{\pi}{ib}}\exp\left(\frac{ia^2}{4b}\right)$.