Exam Quantum Theory, 29 January 2024, 9–12 hours.

1. In this question we address the edge state responsible for the quantum spin Hall effect. Consider the Dirac Hamiltonian

$$\hat{H} = v\hat{p}\sigma_x,$$

which describes the free motion of a massless fermion along the x-axis. The momentum operator $\hat{p} = -i\hbar d/dx$, the Pauli matrix $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and v is a parameter with the dimension of velocity.

- *a)* Explain why this Hamiltonian satisfies time-reversal symmetry.
- *b)* Calculate the energy-momentum relation $E(p)$ (the dispersion relation). Plot it and indicate on which part of the graph the particle moves towards positive x and on which part it moves towards negative x.
- *c)* Suppose we add to the Hamiltonian a potential $V(x)$ times the identity matrix. Explain why this potential cannot cause backscattering, meaning that it cannot cause the particle to reverse its direction of motion.
  Hint: Recall Kramers theorem.

2. Assume that the Hamiltonian $H = H_0 + V$ is the sum of two time-independent Hermitian operators H_0 (called the "free" part) and V (called the "interaction" part). In the socalled "interaction picture" the time-dependent state $\Psi(t)$ and operator A are transformed as

$$\Psi_{\rm I}(t) = e^{iH_0t/\hbar}\Psi(t), \;\; A_{\rm I}(t) = e^{iH_0t/\hbar}Ae^{-iH_0t/\hbar}.$$

- *a)* Explain why the transformation to the interaction picture has no effect on the expectation value $\bar{A}(t) = \langle\Psi(t)|A|\Psi(t)\rangle$ of an operator A in the state $\Psi(t)$, so that we may equally well write $\bar{A}(t) = \langle\Psi_{\rm I}(t)|A_{\rm I}(t)|\Psi_{\rm I}(t)\rangle$.
- *b)* Derive the Heisenberg equation of motion in the interaction picture:

$$i\hbar\frac{d}{dt}A_{\rm I} = [A_{\rm I}, H_0].$$

- *c)* Show, starting from the Schrödinger equation for $\Psi(t)$, that the evolution equation for $\Psi_{\rm I}(t)$ can be written in the form

$$i\hbar\frac{d}{dt}\Psi_{\rm I}(t) = V_{\rm I}(t)\Psi_{\rm I}(t),$$

without any explicit dependence on H_0.

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3. The dimensionless position and momentum operators can be written in terms of the bosonic creation and annihilation operators $\hat{a}^\dagger$ and $\hat{a}$ (with commutator $[\hat{a}, \hat{a}^\dagger] = 1$), as follows:

$$\hat{x} = \sqrt{\tfrac{1}{2}}(\hat{a}^\dagger + \hat{a}), \quad \hat{p} = \sqrt{\tfrac{1}{2}}i(\hat{a}^\dagger - \hat{a}).$$

The vacuum state $|0\rangle$ is defined by $\hat{a}|0\rangle = 0$. Note that the expectation values of $\hat{x}$ and $\hat{p}$ vanish in the vacuum state.

- *a)* Derive the minimal uncertainty relation in the vacuum state,

$$\langle 0|\hat{x}^2|0\rangle\langle 0|\hat{p}^2|0\rangle = \tfrac{1}{4}.$$

The N-particle Fock state $|N\rangle$, normalized to $\langle N|N\rangle = 1$, is defined by $\hat{a}^\dagger\hat{a}|N\rangle = N|N\rangle$, with $N = 1, 2, 3, \ldots$.

- *b)* Derive, starting from this definition, the recursion relation

$$\hat{a}^\dagger|N\rangle = (N+1)^{1/2}|N+1\rangle.$$

- *c)* Explain why the recursion relation implies that the expectation values of $\hat{x}$ and $\hat{p}$ are zero in a Fock state. Then show that a Fock state is not a state of minimal uncertainty, by deriving that

$$\langle N|\hat{x}^2|N\rangle\langle N|\hat{p}^2|N\rangle = \tfrac{1}{4}(2N+1)^2.$$

4. The Bohr-Sommerfeld quantization condition reads

$$\frac{1}{\hbar}\oint p \cdot dq + \gamma = 2\pi n, \quad n = 0, 1, 2, \ldots.$$

We would like to apply this to the periodic cyclotron motion of an electron (charge e, mass m) in a plane perpendicular to a magnetic field B.
Alvaro thinks he knows the answer: The cyclotron orbit is a circle of radius $l_{\rm cycl} = mv/eB$, the kinetic energy is $E = \frac{1}{2}mv^2$, so
$\oint p \cdot dq = mv \times 2\pi l_{\rm cycl} = 4\pi mE/eB$, which gives the quantization
$E_n = \frac{1}{2}\hbar\omega_c(n - \gamma/2\pi)$, with $\gamma = -\pi$ from two turning points.
This is the wrong answer.

- *a)* Which error has Alvaro made?
- *b)* Give the correct calculation.
- *c)* How would the quantization differ if the electrons are massless, as they are in graphene?