EXAM QUANTUM THEORY, 4 MARCH 2025, 15–18 HOURS.

1. Consider the Hamiltonian of a harmonic oscillator with a time dependent frequency $\omega(t)$,

$$H(t) = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \tfrac{1}{2}m\omega^2(t)x^2. \tag{1}$$

- *a)* Is the Hamiltonian still a Hermitian operator? Why or why not?
- *b)* Is the commutator $[H(t_1), H(t_2)]$ of the Hamiltonian at two different times t_1 and t_2 equal to zero or not? Explain your answer with a calculation.
- *c)* A state $\psi(t)$ evolves in time according to the Schrödinger equation:

$$i\hbar\frac{d}{dt}\psi(t) = H(t)\psi(t). \tag{2}$$

  Calculate the time derivative dE/dt of the average energy $E(t) = \langle\psi(t)|H(t)|\psi(t)\rangle$ and relate it to $\langle\psi(t)|x^2|\psi(t)\rangle$.

2. Consider the *bosonic* creation and annihilation operators $a, b, a^\dagger, b^\dagger$. All commutators are zero, except $[a, a^\dagger] = [b, b^\dagger] = 1$. The Hamiltonian is

$$H = a^\dagger a + bb^\dagger + \gamma a^\dagger b^\dagger + \gamma ab, \tag{3}$$

for some real constant γ. The Boboliubov transformation expresses the annihilation operators a, b in terms of new operators c, d given by

$$c = a\cosh\lambda + b^\dagger\sinh\lambda, \quad d = a^\dagger\sinh\lambda + b\cosh\lambda, \tag{4}$$

for some real constant λ.

- *a)* Check that the operators c, d and $c^\dagger, d^\dagger$ still satisfy the bosonic commutation relations.*
- *b)* Chose λ such that $\tanh 2\lambda = \gamma$. Show that

$$H = \frac{1}{\cosh 2\lambda}(c^\dagger c + dd^\dagger). \tag{5}$$

- *c)* Explain why the spectrum of H consists of equidistant levels. What is the level spacing?

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*Recall the properties of the hyperbolic sines and cosines: $\cosh^2 x - \sinh^2 x = 1$, $\cosh 2x = \cosh^2 x + \sinh^2 x$, $\sinh 2x = 2\sinh x\cosh x$.

3. The Hamiltonian H has eigenvalues E_n, $n = 0, 1, 2, \ldots$, with corresponding eigenfunctions Φ_n. The ground state Φ_0 has the lowest eigenvalue E_0. An arbitrary wave function ψ can be expanded in the basis of eigenfunctions, $\psi = \sum_n \alpha_n \Phi_n$, with complex coefficients α_n.

- *a)* Express the inner product $\langle\psi|\psi\rangle$ in terms of these expansion coefficients.

  The variational theorem says that

$$\frac{\langle\psi|H|\psi\rangle}{\langle\psi|\psi\rangle} \geq E_0.$$

- *b)* Prove this theorem. The one-dimensional harmonic oscillator has Hamiltonian

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dx} + \tfrac{1}{2}m\omega^2 x^2.$$

  For a given parameter $a > 0$ we approximate the ground state wave function by

$$\Phi_a(x) \approx C \exp\left(-\frac{x^2}{2a}\right),$$

  with normalization constant $C = (2\pi a)^{-1/2}$.

- *c)* Explain how you would use the variational theorem to approximate the ground state energy E_0. (You don't have to actually carry out the calculation, just explain which steps you would take.)

4. A particle (charge q, mass m) in a magnetic field $\vec{B}(\vec{r}) = \nabla \times \vec{A}(\vec{r})$ has Hamiltonian

$$H = \frac{1}{2m}(\vec{p} - q\vec{A})^2, \quad \text{with} \quad \vec{p} = -i\hbar\nabla.$$

- *a)* Derive the Heisenberg equation of motion for the position operator $\vec{r}$, to obtain an expression for the velocity operator $\vec{v}$.

  We now investigate the effect of a gauge transformation of the vector potential, $\vec{A}'(\vec{r}) = \vec{A}(\vec{r}) + \nabla\chi(\vec{r})$, for some arbitrary function $\chi(\vec{r})$. The Hamiltonian with $\vec{A}$ replaced by $\vec{A}'$ is denoted by H'.

- *b)* Verify that H and H' are related by

$$H' = \exp(iq\chi/\hbar) H \exp(-iq\chi/\hbar).$$

- *c)* Explain why this relation between H and H' expresses the fact that the vector potentials $\vec{A}$ and $\vec{A}'$ describe the same physical system.