EXAM QUANTUM THEORY, 12 JANUARY 2026, 9–12 HOURS.

1. The Hellmann-Feynman theorem says that, if the Hamiltonian $H(\lambda)$ depends on a parameter λ, and $E_n(\lambda)$ is an eigenvalue with eigenstate $\psi_n(\lambda)$, then

$$\frac{d}{d\lambda}E_n(\lambda) = \langle\psi_n(\lambda)|\frac{dH(\lambda)}{d\lambda}|\psi_n(\lambda)\rangle.$$

- *a)* Give a proof of this theorem.
- *b)* If the Hamiltonian is independent of the x-coordinate, the momentum p in the x-direction is conserved and the energy $E(p)$ is a function of p. Use the Hellmann-Feynman theorem to explain how the expectation value $\langle v\rangle$ of the velocity can be obtained from $E(p)$.

  Consider the harmonic oscillator Hamiltonian (kinetic energy T, potential energy V, mass m, oscillator frequency ω),

$$H = T + V,\ \ T = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2},\ \ V = \tfrac{1}{2}m\omega^2x^2.$$

  The eigenvalues are $E_n = (n + 1/2)\hbar\omega$, $n = 0, 1, 2\ldots$.

- *c)* Use the Hellmann-Feynman theorem to prove the equipartitioning of kinetic and potential energy: $\langle\psi_n|T|\psi_n\rangle = \langle\psi_n|V|\psi_n\rangle$.
  Hint: Calculate dE_n/dm.

2. The wave function of an electron has two components, $\psi_\uparrow(x)$ (spin up) and $\psi_\downarrow(x)$ (spin down). The corresponding Hamiltonian is a 2×2 matrix,

$$H = \begin{pmatrix} 0 & -i\hbar v d/dx \\ -i\hbar v d/dx & 0 \end{pmatrix},$$

  where v is a real constant (not an operator). An eigenstate at energy E satisfies

$$H\begin{pmatrix}\psi_\uparrow(x)\\ \psi_\downarrow(x)\end{pmatrix} = E\begin{pmatrix}\psi_\uparrow(x)\\ \psi_\downarrow(x)\end{pmatrix}.$$

- *a)* Xavier objects that this Hamiltonian is not Hermitian. What do you think?
- *b)* Does this Hamiltonian satisfy time reversal symmetry or not? Explain your answer.
- *c)* Eigenstates are plane waves $\propto e^{ikx}$, for some wave number k. How does the energy depend on k? Illustrate your finding with a plot of $E(k)$.

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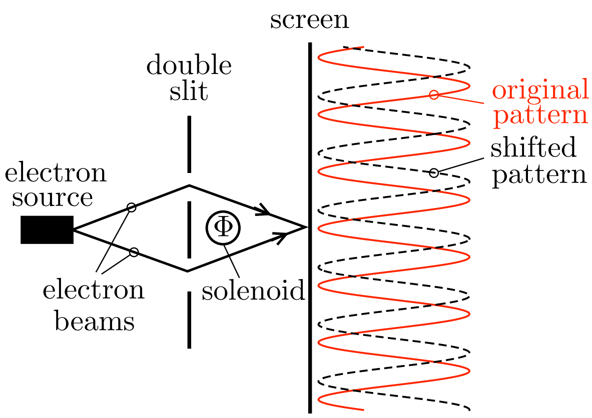

3. Consider a solenoid (*spoel*) of radius R enclosing a magnetic flux Φ. Outside the solenoid, the magnetic field $\vec{B} = 0$.

- *a)* Is it possible to choose a gauge such that the vector potential $\vec{A} = 0$ outside the solenoid? Explain your answer.

  The solenoid is placed between two slits, in front of a screen, see the figure. You may assume that the electrons that move from source to screen via the slits do not enter the solenoid. You may also assume that the electrons only move in the two-dimensional plane of the figure, perpendicular to the solenoid.

- *b)* Give an expression for the wave amplitude on the screen in the semiclassical WKB approximation, as a sum over classical electron trajectories from source to screen.

  The interference pattern on the screen of electrons transmitted through the slits shifts when Φ is varied. The original interference pattern is recovered when Φ is increased by an amount $\Delta\Phi$.

- *c)* Calculate $\Delta\Phi$. Verify that your answer does not depend on the gauge you chose for the vector potential.

4. The bosonic annihilation operator $\hat{a}$ and creation operator $\hat{a}^\dagger$ can be used to describe the state of photons at a given frequency. A laser emits photons in a socalled coherent state, given by

$$|\beta\rangle = e^{-|\beta|^2/2} e^{\beta \hat{a}^\dagger} |0\rangle.$$

   Here β is a complex number and $|0\rangle$ is the vacuum state.

- *a)* Show that the coherent state is an eigenstate of the annihilation operator: $\hat{a}|\beta\rangle = \beta|\beta\rangle$.

- *b)* Show that two coherent states $|\alpha\rangle$ and $|\beta\rangle$ are not orthogonal, by deriving that $|\langle\alpha|\beta\rangle|^2 = \exp(-|\alpha - \beta|^2)$.

- *c)* Calculate the first two moments of the photon number: $\bar{n} = \langle\beta|\hat{a}^\dagger\hat{a}|\beta\rangle$ and $\overline{n^2} = \langle\beta|(\hat{a}^\dagger\hat{a})^2|\beta\rangle$, and the variance $\operatorname{var} n = \overline{n^2} - (\bar{n})^2$.