Exam Quantum Theory, 18 February 2026, 15.15–18.15 hours.

1. In a one-dimensional system, the operator T_a that translates over a distance a can be defined by its action on a wave function $\psi(x)$: $T_a\psi(x) = \psi(x+a)$.

- *a)* Recall the definition of a unitary operator and prove that T_a is unitary.

- *b)* What are the eigenvalues of T_a ?

- *c)* The Hamiltonian $H = p^2/2m + V(x)$ commutes with T_a if $V(x+a) = V(x)$. Assume that this is the case and prove that the wave function $\psi(x)$ of any nondegenerate eigenstate of H should satisfy $\psi(x+a) = \lambda\psi(x)$ for all x, for some complex coefficient λ.

2. The *squeezed vacuum* for photons is the state $|s\rangle \equiv S(s)|0\rangle$ obtained by acting on the vacuum state $|0\rangle$ with the squeeze operator

$$S(s) = \exp\left(\tfrac{1}{2}s(aa - a^\dagger a^\dagger)\right).$$

Here s is a real number and $a, a^\dagger$ are bosonic annihilation and creation operators (commutator $[a, a^\dagger] = 1$).

- *a)* Is $S(s)$ unitary? Is it Hermitian?

In what follows you may use the identity

$$S^\dagger(s)aS(s) = a\cosh s - a^\dagger\sinh s.$$

- *b)* The momentum operator is $p = 2^{-1/2}i(a^\dagger - a)$ (in dimensionless units). Calculate the variance $\Delta p^2 = \langle s|p^2|s\rangle - \langle s|p|s\rangle^2$ of the momentum in the squeezed vacuum state.

- *c)* For $s \to -\infty$ the variance of the momentum goes to zero. Does this contradict the uncertainty principle? Please explain.

3. Consider the Dirac Hamiltonian

$$H = \begin{pmatrix} -i\hbar v \, d/dx & 0 \\ 0 & i\hbar v \, d/dx \end{pmatrix},$$

which describes the free motion of a massless fermion along the x-axis. Here v is a real parameter with the dimension of velocity.

- *a)* Explain why this *complex* Hamiltonian is Hermitian and explain whether or not H satisfies time-reversal symmetry

- *b)* Search for eigenstates of H of the form $e^{ikx}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ or $e^{ikx}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, for some real parameter k. Plot the k-dependence $E(k)$ of the energy eigenvalue and indicate on which part of the graph the particle moves towards positive x and on which part it moves towards negative x.

- *c)* Suppose we add to the Hamiltonian a term

$$M = \begin{pmatrix} 0 & \mu \\ \mu & 0 \end{pmatrix},$$

for some real constant μ. Show that an energy gap opens in the spectrum around $E = 0$.
Hint: Try squaring the Hamiltonian.

4. A particle (charge q, mass m) in a magnetic field $\vec{B}(\vec{r}) = \nabla \times \vec{A}(\vec{r})$ has Hamiltonian

$$H = \frac{1}{2m}(\vec{p} - q\vec{A})^2, \quad \text{with } \vec{p} = -i\hbar\nabla.$$

- *a)* Derive the Heisenberg equation of motion for the position operator $\vec{r}$, to obtain an expression for the velocity operator $\vec{v}$.

  We now investigate the effect of a gauge transformation of the vector potential, $\vec{A}'(\vec{r}) = \vec{A}(\vec{r}) + \nabla\chi(\vec{r})$, for some arbitrary function $\chi(\vec{r})$. The Hamiltonian with $\vec{A}$ replaced by $\vec{A}'$ is denoted by H'.

- *b)* Verify that H and H' are related by

$$H' = \exp(iq\chi/\hbar)H\exp(-iq\chi/\hbar).$$

- *c)* Explain why this relation between H and H' expresses the fact that the vector potentials $\vec{A}$ and $\vec{A}'$ describe the same physical system.