1 Fundamental concepts

1.1 Bra-ket notation, delta function, position and momentum representation

a)

• $\langle 1|1\rangle = 1 = \langle 2|2\rangle$, $\langle 1|2\rangle = 0$, $\Rightarrow$

$$\begin{aligned}
\langle \psi_+|\psi_+\rangle &= \tfrac{1}{2}(\langle 1| - i\langle 2|)(|1\rangle + i|2\rangle) = 1,\\
\langle \psi_-|\psi_-\rangle &= \tfrac{1}{2}(\langle 1| + i\langle 2|)(|1\rangle - i|2\rangle) = 1,\\
\langle \psi_-|\psi_+\rangle &= \tfrac{1}{2}(\langle 1| + i\langle 2|)(|1\rangle + i|2\rangle) = 0.
\end{aligned}$$

• $\hat{A}\hat{B} = |\psi_+\rangle\langle\psi_+|$, $\hat{B}\hat{A} = |\psi_-\rangle\langle\psi_-|$.

$$(|\psi_+\rangle\langle\psi_+|)^\dagger = |\psi_+\rangle\langle\psi_+|,\ \ (|\psi_+\rangle\langle\psi_+|)^2 = |\psi_+\rangle\langle\psi_+|.$$

• $\hat{A}^\dagger = |\psi_-\rangle\langle\psi_+| = \hat{B},\ \ \hat{B}^\dagger = \hat{A}$.
• $\mathrm{Tr}\,\hat{A} = \langle\psi_-|\psi_+\rangle = 0 = \mathrm{Tr}\,\hat{B}$; $\mathrm{Tr}\,\hat{A}\hat{B} = \langle\psi_+|\psi_+\rangle = 1$, $\mathrm{Tr}\,\hat{B}\hat{A} = \langle\psi_-|\psi_-\rangle = 1$.
• The matrix elements $H_{nm} = \langle n|\hat{H}|m\rangle$ are $H_{11} = 1,\ H_{22} = -1,\ H_{12} = i,\ H_{21} = -i$, in matrix form

$$H = \begin{pmatrix} 1 & i \\ -i & -1 \end{pmatrix}.$$

The usual linear algebra calculation of eigenvalues and eigenvectors gives $E_\pm = \pm\sqrt{2}$, $|\psi_\pm\rangle = c_\pm[i(1\pm\sqrt{2})|1\rangle + |2\rangle]$, with normalization constant $c_\pm = (4\pm 2\sqrt{2})^{-1/2}$.
The trace of H is zero, so the two eigenvalues must sum to zero.
• $(\hat{A}+\hat{B})^2 = |\psi_+\rangle\langle\psi_+| + |\psi_-\rangle\langle\psi_-| = \hat{1}$. So the eigenvalues squared of $\hat{A}+\hat{B}$ equal 1, while their sum vanishes (since $\hat{A}$ and $\hat{B}$ have trace zero), hence the eigenvalues are ± 1.

b)

$$\int_{-\infty}^{\infty} dy\, f(y)\frac{\partial}{\partial y}\delta(y-x) = -\int_{-\infty}^{\infty} dy\, f'(y)\delta(y-x) = -f'(x), \tag{1}$$

$$\int_{-\infty}^{\infty} dx\, \delta(ax) = \frac{1}{a}\int_{-\infty}^{\infty} d(ax)\,\delta(ax) = \frac{1}{a}, \tag{2}$$

$$\int_{-\infty}^{\infty} dx\, f(x)\delta(x^2-a^2) = \int_{-\infty}^{\infty} dx\, f(x)\left[\frac{\delta(x-a)}{x+a} + \frac{\delta(x+a)}{x-a}\right] = \frac{f(a)-f(-a)}{2a}. \tag{3}$$

c)

$$\hat{1}|x'\rangle = \int_{-\infty}^{\infty} dx\,|x\rangle\langle x|x'\rangle = |x'\rangle \Rightarrow \langle x|x'\rangle = \delta(x-x'), \tag{4}$$

$$\langle x|\hat{q}|x'\rangle = x'\langle x|x'\rangle = x\delta(x-x'), \tag{5}$$

$$\begin{aligned}\langle x|\hat{p}|x'\rangle &= \int dp'\,\langle x|\hat{p}|p'\rangle\langle p'|x'\rangle = \frac{1}{2\pi\hbar}\int dp'\,p' e^{ip'x/\hbar}e^{-ip'x'/\hbar}\\ &= -i\frac{\partial}{\partial x}\int\frac{dp'}{2\pi}e^{ip'(x-x')/\hbar} = -i\hbar\frac{\partial}{\partial x}\delta(x-x') = \langle x'|\hat{p}|x\rangle^*, \end{aligned}\tag{6}$$

$$\begin{aligned}\langle x|\hat{q}\hat{p}-\hat{p}\hat{q}|x'\rangle &= (x-x')\langle x|\hat{p}|x'\rangle = -i\hbar(x-x')\frac{\partial}{\partial x}\delta(x-x')\\ &= -i\hbar\frac{\partial}{\partial x}[(x-x')\delta(x-x')] + i\hbar\delta(x-x')\frac{\partial}{\partial x}(x-x') = i\hbar\delta(x-x'). \end{aligned}\tag{7}$$

1.2 Heisenberg equation of motion and Ehrenfest theorem

a) $[\hat{q},\hat{p}^2/2m] = i(\hbar/m)\hat{p},\ \hat{p}(t) = \hat{p}(0)$

$$\Rightarrow\ \hat{q}(t) = \hat{q}(0) + (t/m)\hat{p}(0),\ \ [\hat{q}(t),\hat{q}(0)] = -i\hbar(t/m).$$

b) $\langle\hat{q}(t)\rangle = \langle\hat{q}(0)\rangle + (t/m)\langle\hat{p}(0)\rangle$.

1.3 Hellmann-Feynman theorem

a) $E_n = \langle\psi_n|H|\psi_n\rangle$, hence

$$\begin{aligned} dE_n/d\lambda &= \langle\psi_n|\frac{d\hat{H}}{d\lambda}|\psi_n\rangle + \langle\frac{d\psi_n}{d\lambda}|\hat{H}|\psi_n\rangle + \langle\psi_n|\hat{H}|\frac{d\psi_n}{d\lambda}\rangle\\ &= \langle\psi_n|\frac{d\hat{H}}{d\lambda}|\psi_n\rangle + E_n\frac{d}{d\lambda}\langle\psi_n|\psi_n\rangle = \langle\psi_n|\frac{d\hat{H}}{d\lambda}|\psi_n\rangle, \end{aligned}$$

since $\langle\psi_n|\psi_n\rangle = 1$.

b) The velocity operator along the wave guide is $\hat{v} = d\hat{H}/dp$.

1.4 Uncertainty relation

a) $\langle\psi|\hat{T}^\dagger\hat{T}|\psi\rangle = |\hat{T}\Psi_n|^2 \geqslant 0$.

b) $\langle\psi|\hat{A}^2+\omega^2\hat{B}^2-i\omega\hat{B}\hat{A}+i\omega\hat{A}\hat{B}|\psi\rangle \geqslant 0$.

c) Choose $\omega=\frac{1}{2}\langle\hat{C}\rangle/\langle\Delta\hat{B}\rangle^2$.

d) $(\Delta x)^2=\int dx\,|x\psi|^2=1/(4\alpha)$, $(\Delta p)^2=\hbar^2\int dx\,|d\psi/dx|^2=\hbar^2\alpha$.