# 2 Symmetries

## 2.1 Inversion symmetry

*a)* on the one hand $\mathscr{P}[\hat{q},\hat{p}]\mathscr{P}^{-1} = \mathscr{P} i\hbar\mathscr{P}^{-1}$ on the other hand $\mathscr{P}[\hat{q},\hat{p}]\mathscr{P}^{-1} = [\hat{q},\hat{p}] = i\hbar$, so $\mathscr{P} i\mathscr{P}^{-1} = i$.

*b)* $\mathscr{P}^2 = e^{i\phi}\hat{1}$, redefining $\mathscr{P}' = e^{-i\phi/2}\mathscr{P}$ has no effect on the inversion operation. Eigenvalues of $\mathscr{P}'$ are $\pm 1$.

*c)* $\hat{H}\mathscr{P}|\Psi\rangle = \mathscr{P}\hat{H}|\Psi\rangle = E\mathscr{P}|\Psi\rangle$ and because the eigenstate $|\Psi\rangle$ of $\hat{H}$ with eigenvalue $E$ is nondegenerate the two states $\mathscr{P}|\Psi\rangle$ and $|\Psi\rangle$ must be linearly related: $\mathscr{P}|\Psi\rangle = \lambda|\Psi\rangle$ and $\mathscr{P}^2|\Psi\rangle = |\Psi\rangle = \lambda^2|\Psi\rangle$, so $\lambda = \pm 1$.
$\langle\Psi|\hat{\boldsymbol{q}}_n|\Psi\rangle = \langle\mathscr{P}\Psi|\mathscr{P}\hat{\boldsymbol{q}}_n\mathscr{P}^{-1}|\mathscr{P}\Psi\rangle = -\langle\Psi|\hat{\boldsymbol{q}}_n|\Psi\rangle \Rightarrow \langle\Psi|\hat{\boldsymbol{q}}_n|\Psi\rangle = 0$

## 2.2 Time-reversal symmetry

*a)* on the one hand $\mathscr{T}[\hat{q},\hat{p}]\mathscr{T}^{-1} = \mathscr{T} i\hbar\mathscr{T}^{-1}$ on the other hand $\mathscr{T}[\hat{q},\hat{p}]\mathscr{T}^{-1} = -[\hat{q},\hat{p}] = -i\hbar$, so $\mathscr{T} i\mathscr{T}^{-1} = -i$.

*b)* $\mathscr{T}^2 = e^{i\phi}\hat{1} = \hat{U}\hat{U}^*$. Multiply with $U^{-1}$ from the left and with $U$ from the right, and conclude that also $\hat{U}^*U = e^{i\phi}\hat{1} = (\hat{U}\hat{U}^*)^*$. Therefore $e^{i\phi} = e^{-i\phi} \Rightarrow e^{i\phi} = \pm 1$.

*c)* $\mathscr{T}^2 = \sigma_2\sigma_2^* = -1$

*d)* $\hat{H}$ remains invariant if momentum and spin are inversed.

*e)* $\hat{H}\mathscr{T}|\psi\rangle = \mathscr{T}\hat{H}|\psi\rangle = E\mathscr{T}|\psi\rangle$ so if the eigenstate $|\psi\rangle$ of $\hat{H}$ with eigenvalue $E$ is nondegenerate then the two states $\mathscr{T}|\psi\rangle$ and $|\psi\rangle$ must be linearly related:
$|\psi\rangle = \lambda\mathscr{T}|\psi\rangle = \lambda\mathscr{T}\lambda\mathscr{T}|\psi\rangle = -|\lambda|^2|\psi\rangle$ — contradiction

## 2.3 Galilean invariance

*a)* $[\hat{G},\hat{x}] = ivt,\ \ [e^{i\hat{G}},\hat{x}] = -vte^{i\hat{G}} \Rightarrow e^{i\hat{G}}\hat{x}e^{-i\hat{G}} = \hat{x} - vt$

$$\text{proof: } [G^n,x] = nivtG^{n-1} \Rightarrow [e^{iG},x] = \left[\sum_{n=0}^{\infty}\frac{(iG)^n}{n!},x\right] = (-vt)\sum_{n=1}^{\infty}\frac{(iG)^{n-1}}{(n-1)!} = -vte^{iG}$$

similarly, $[\hat{G},\hat{p}] = imv,\ \ [e^{i\hat{G}},\hat{p}] = -mve^{i\hat{G}} \Rightarrow e^{i\hat{G}}\hat{p}e^{-i\hat{G}} = \hat{p} - mv$

*b)*

$$\frac{d}{dt}\hat{G} = \frac{\partial}{\partial t}\hat{G} - \frac{i}{\hbar}[\hat{G}, \hat{H}] = -\frac{\hat{p}v}{\hbar} - \frac{iv}{2\hbar^2}[\hat{x}, \hat{p}^2] = 0.$$