3 Fermions and bosons

3.1 Second quantization

a)

$$\begin{aligned}
\hat{U} &= \int_V d\vec{r}\ \hat{n}(\vec{r})\, u(\vec{r}) \\
&= \frac{1}{V}\int_V d\vec{r}\ \sum_{\vec{k}}\sum_{\vec{k}'} e^{-i(\vec{k}-\vec{k}')\cdot\vec{r}}\ \hat{a}^{\dagger}_{\vec{k}}\hat{a}_{\vec{k}'} u(\vec{r}) \\
&= \frac{1}{\sqrt{V}}\sum_{\vec{k}}\sum_{\vec{k}'} \hat{a}^{\dagger}_{\vec{k}}\hat{a}_{\vec{k}'} u_{\vec{k}-\vec{k}'} \\
&= \sum_{\vec{q}} \frac{u_{\vec{q}}}{\sqrt{V}} \sum_{\vec{m}} \hat{a}^{\dagger}_{\vec{m}+\vec{q}}\, \hat{a}_{\vec{m}}.
\end{aligned}$$

b)

$$\begin{aligned}
v(\vec{k}\vec{\ell},\vec{m}\vec{n}) &= \frac{1}{V^2}\int d\vec{r}_1 d\vec{r}_2\, e^{-i(\vec{k}\cdot\vec{r}_1+\vec{\ell}\cdot\vec{r}_2-\vec{m}\cdot\vec{r}_1-\vec{n}\cdot\vec{r}_2)}\, v(r_{12}) \\
&= \frac{1}{\sqrt{V}}\, v_{\vec{k}-\vec{m}}\, \frac{1}{V}\int d\vec{r}_2\, e^{-i(\vec{k}+\vec{\ell}-\vec{m}-\vec{n})\cdot\vec{r}_2} \\
&= \delta_{\vec{k}+\vec{\ell},\vec{m}+\vec{n}}\, \frac{1}{\sqrt{V}}\, v_{\vec{k}-\vec{m}}.
\end{aligned}$$

c)

$$\begin{aligned}
\hat{H}^{(\mathrm{int})} &= \frac{1}{2}\sum_{i\neq j} v\left(|\hat{\vec{r}}^{(i)} - \hat{\vec{r}}^{(j)}|\right) \\
&= \frac{1}{2}\int d\vec{r}_1 d\vec{r}_2\, n(\vec{r}_1)\, n(\vec{r}_2)\, v(r_{12}) \qquad [\text{assume } v(0)=0] \\
&= \frac{1}{2V^2}\int d\vec{r}_1 d\vec{r}_2\, v(r_{12}) \sum_{\vec{k}}\sum_{\vec{m}} e^{-i(\vec{k}-\vec{m})\cdot\vec{r}_1}\ \hat{a}^{\dagger}_{\vec{k}}\hat{a}_{\vec{m}} \sum_{\vec{k}'}\sum_{\vec{n}} e^{-i(\vec{k}'-\vec{n})\cdot\vec{r}_2}\ \hat{a}^{\dagger}_{\vec{k}'}\hat{a}_{\vec{n}} \\
&= \sum_{\vec{q}} \frac{v_{\vec{q}}}{2\sqrt{V}} \sum_{\vec{m},\vec{n}} \hat{a}^{\dagger}_{\vec{m}+\vec{q}}\hat{a}_{\vec{m}}\, \hat{a}^{\dagger}_{\vec{n}-\vec{q}}\, \hat{a}_{\vec{n}},\ \text{with } \vec{q}=\vec{k}-\vec{m}.
\end{aligned}$$

3.2 Coherent states

a)

$$[a, e^{\beta a^{\dagger}}] = \beta e^{\beta a^{\dagger}} \Rightarrow a e^{\beta a^{\dagger}}|0\rangle = \beta e^{\beta a^{\dagger}}|0\rangle.$$

Normalization:

$$\langle\beta|\beta\rangle = e^{-|\beta|^2/2}\langle 0|e^{\beta^* a}|\beta\rangle = e^{|\beta|^2/2}\langle 0|\beta\rangle = 1.$$

b)

$$\langle\beta|(a^\dagger)^p a^q|\beta\rangle = (\beta^*)^p\beta^q\langle\beta|\beta\rangle = (\beta^*)^p\beta^q.$$

c)

$$|n\rangle = \frac{1}{\sqrt{n!}}(a^\dagger)^n|0\rangle,\;\; P(n) = |\langle n|\beta\rangle|^2 = \frac{1}{n!}|\langle 0|a^n|\beta\rangle|^2 = \frac{|\beta|^{2n}}{n!}|\langle 0|\beta\rangle|^2 = e^{-|\beta|^2}\frac{|\beta|^{2n}}{n!}.$$

d)

$$\begin{aligned}
&\langle\beta|x|\beta\rangle = 2^{-1/2}(\beta+\beta^*),\;\; \langle\beta|p|\beta\rangle = 2^{-1/2}i(\beta^*-\beta),\\
&\langle\beta|x^2|\beta\rangle = \tfrac{1}{2}(\beta^2+\beta^{*2}+2\beta\beta^*+1) \Rightarrow \langle(\Delta x)^2\rangle = 1/2,\\
&\langle\beta|p^2|\beta\rangle = -\tfrac{1}{2}(\beta^2+\beta^{*2}-2\beta\beta^*-1) \Rightarrow \langle(\Delta p)^2\rangle = 1/2.
\end{aligned}$$

3.3 Bogoliubov transformations

The Bogoliubov transformation is a linear transformation of the creation and annihilation operators that preserves their commutation relation. We examine this for a pair of fermionic creation and annihilation operators $\hat{a}^\dagger_\alpha$ and $\hat{a}_\alpha$, with $\alpha = \pm$ labeling spin and/or momentum. The commutation relations are

$$\hat{a}_\alpha\hat{a}_{\alpha'} + \hat{a}_{\alpha'}\hat{a}_\alpha = 0 \text{ and } \hat{a}_\alpha\hat{a}^\dagger_{\alpha'} + \hat{a}^\dagger_{\alpha'}\hat{a}_\alpha = \delta_{\alpha\alpha'}\hat{1}.$$

Consider the linear transformation

$$\hat{b}_\alpha = u_\alpha\hat{a}_\alpha - v_\alpha\hat{a}^\dagger_{-\alpha},$$

with real coefficients u_α and v_α.

a)

$$\hat{b}^\dagger_\alpha = u_\alpha\hat{a}^\dagger_\alpha - v_\alpha\hat{a}_{-\alpha}.$$

b)

$$\begin{aligned}
&\{\hat{b}_\alpha\hat{b}_{\alpha'}\} = \{u_\alpha\hat{a}^\dagger_\alpha - v_\alpha\hat{a}_{-\alpha}, u_{\alpha'}\hat{a}^\dagger_{\alpha'} - v_{\alpha'}\hat{a}_{-\alpha'}\} = -u_\alpha v_{-\alpha} - v_\alpha u_{-\alpha} = 0,\\
&\{\hat{b}_\alpha\hat{b}^\dagger_{\alpha'}\} = \{u_\alpha\hat{a}^\dagger_\alpha - v_\alpha\hat{a}_{-\alpha}, u_{\alpha'}\hat{a}_{\alpha'} - v_{\alpha'}\hat{a}^\dagger_{-\alpha'}\} = \delta_{\alpha\alpha'}(u^2_\alpha + v^2_\alpha) = \delta_{\alpha\alpha'}.
\end{aligned}$$

c)

$$\hat{b}_+ = u\hat{a}_+ - v\hat{a}_-^\dagger,\ \ \hat{b}_- = u\hat{a}_- + v\hat{a}_+^\dagger, \Rightarrow a_+ = ub_+ + vb_-^\dagger,\ \ a_- = ub_- - vb_+^\dagger.$$

d) the terms $\propto \Delta$ create or annihilate a pair of electrons.

e)

$$\begin{aligned}\hat{H} &= \xi[(ub_+^\dagger + vb_-)(ub_+ + vb_-^\dagger) + (ub_-^\dagger - vb_+)(ub_- - vb_+^\dagger)] + \\ &\quad + \Delta[(ub_+^\dagger + vb_-)(ub_-^\dagger - vb_+) + (ub_- - vb_+^\dagger)(ub_+ + vb_-^\dagger)] \\ &= [\xi(u^2 - v^2) - 2\Delta uv](b_+^\dagger b_+ + b_-^\dagger b_-) \\ &\quad + [2\xi uv + \Delta(u^2 - v^2)](b_+^\dagger b_-^\dagger + b_- b_+).\end{aligned}$$

for the special choice of u, v one has

$$4u^2v^2 = \frac{\Delta^2}{\xi^2 + \Delta^2},\ \ u^2 - v^2 = \frac{\xi}{\sqrt{\xi^2 + \Delta^2}}.$$

Choose $uv < 0$, then $2\xi uv + \Delta(u^2 - v^2) = 0$ and

$$\hat{H} = (\xi^2 + \Delta^2)^{1/2}\,(b_+^\dagger b_+ + b_-^\dagger b_-).$$

There is an excitation gap Δ.

3.4 Majorana fermions

Following Kitaev, consider spin-polarized fermions on a chain of N sites with Hamiltonian

$$\hat{H} = \sum_{j=1}^{N-1}\left[t(\hat{a}_j^\dagger \hat{a}_{j+1} + \hat{a}_{j+1}^\dagger \hat{a}_j) + \Delta(\hat{a}_j \hat{a}_{j+1} + \hat{a}_{j+1}^\dagger \hat{a}_j^\dagger)\right] - \mu \sum_{j=1}^{N} \hat{a}_j^\dagger \hat{a}_j,$$

where t is the hopping amplitude between neighbouring sites, μ is the chemical potential, and Δ is the pair potential.

We make the transformation

$$\gamma_{2j-1} = a_j + a_j^\dagger \text{ and } \gamma_{2j} = -i(a_j - a_j^\dagger).$$

indicated in the figure. The γ operators are called "Majorana operators" and the quasiparticles they represent are called "Majorana fermions".

a) $\gamma_n^\dagger = \gamma_n$; creation = annihilation *b)* $\{\gamma_n, \gamma_m\} = 2\delta_{nm}$, $\gamma_n^2 = 1$.

c)

$$\hat{H} = 2i \sum_{j=1}^{N-1} \left[(\Delta + t)\gamma_{2j-1}\gamma_{2j+2} + (\Delta - t)\gamma_{2j}\gamma_{2j+1} \right] - 2i\mu \sum_{j=1}^{N} \gamma_{2j-1}\gamma_{2j} - 2N\mu.$$

d) When $\Delta = -t > 0$, $\mu = 0$ one has $\hat{H} = 4i\Delta \sum_{j=1}^{N-1} \gamma_{2j}\gamma_{2j+1}$. The operators γ_1 and γ_{2N} do not appear, corresponding to the fermion operator $b = \gamma_1 + i\gamma_{2N}$, split over the two end points of the chain.