4 Quantum electrodynamics

4.1 Gauge transformation, Aharonov-Bohm effect & Byers-Yang theorem

a) take $U = \exp[ie\chi(\boldsymbol{r})/\hbar]$

$$
\begin{aligned}
U[\boldsymbol{p} - e\boldsymbol{A}]^2 U^{-1} &= (U[\boldsymbol{p} - e\boldsymbol{A}]U^{-1})^2 = \big(\exp[ie\chi(\boldsymbol{r})/\hbar][(-i\hbar)\nabla - e\boldsymbol{A}]\exp[-ie\chi(\boldsymbol{r})/\hbar]\big)^2 \\
&= [\boldsymbol{p} - e\boldsymbol{A} - e\nabla\chi(\boldsymbol{r})]^2
\end{aligned}
$$

A ring enclosing a line of magnetic flux Φ at the origin has vector potential $\boldsymbol{A}(r,\phi) = (\Phi/2\pi r)\hat{\phi}$ in polar coordinates. Because $\boldsymbol{B} = 0$ for all $r \neq 0$, we can perform a gauge transformation with $\chi(r,\phi) = (\Phi/2\pi)\phi$ that removes the vector potential from the ring, $\boldsymbol{A}' = \boldsymbol{A} + \nabla\chi = 0$ for $r \neq 0$.

b) The gauge transformed wave function $\psi' = U\psi = \exp[i(e/\hbar)(\Phi/2\pi)\phi]\psi$ is no longer single-valued, so this gauge transformation is forbidden, except when $(e/\hbar)\Phi/2\pi = e\Phi/h$ is an integer. So the physical properties (conductance) of this ring may still depend on the enclosed flux modulo h/e, even when the magnetic field by the electrons vanishes everywhere. That is the Aharonov-Bohm effect.

c) The gauge transformation preserves a single-valued wave function when $(e/\hbar)(\Phi/2\pi)\phi = (e/\hbar)(\Phi/2\pi)(\phi + 2\pi)$ modulo 2π, so when $(e/\hbar)\Phi$ is a multiple of 2π, or equivalently when Φ is a multiple of h/e. So we can always add a multiple of h/e to the flux, and physical properties cannot change.

4.2 Persistent currents

a)

$$
\frac{dE_0}{d\Phi} = \left\langle \frac{d\hat{H}}{d\Phi} \right\rangle_0 = -\frac{e}{mL}\langle \hat{p} - eA\rangle_0 = -\langle \hat{I}\rangle_0 = -I_0
$$

b) take $U = \exp(2\pi ix/L)$

$$
UHU^{-1} = \frac{1}{2m}(p - 2\pi\hbar/L - e\Phi/L)^2 + V
$$

so $H(\Phi)$ and $H(\Phi + h/e)$ are related by a unitary transformation and hence have the same spectrum (so the same I_0).

c) $\psi(x) = L^{-1/2}e^{ikx}$ is eigenstate of H at energy $E(k,\Phi) = (\hbar k - e\Phi/L)^2/2m$; periodic boundary conditions require that $kL = 2\pi n$, $n = 0, \pm 1, \pm 2, \ldots$. Take $\Phi \in (-h/2e, h/2e)$, then the ground state has $n = 0$: $E_0 = e^2\Phi^2/2mL^2$, hence $I_0 = -e^2\Phi/mL^2$, maximal for $|\Phi| = h/2e$, equal to $|I_0| = \pi e\hbar/mL^2$.

4.3 Casimir effect

a) A plane wave e^{ikx} with periodic boundary conditions at $x = 0$ and $x = L$ must have a wave vector k that is a multiple of $2\pi/L$. The sum $\sum_n f(k_n)$ with $k_n = 2\pi n/L$ can be converted into an integral $(L/2\pi)\int dk\, f(k)$. In three dimensions this gives a factor $V(2\pi)^{-3}$, with $V = L^3$ the volume of the system, which is absorbed in the definition of $\mathscr{E}_0$.

b) for $d \to \infty$ the sum over n can be converted into an integral over a continuous variable n; changing integration variables from n to $n\pi/d$ gives a factor d/π which combines with the factor $1/2d$ to give the factor $1/2\pi$ in $\mathscr{E}_0$.

c) First transform the expression for $\mathscr{E}_{\text{plates}}$, in the following steps:

$$\begin{aligned}
\mathscr{E}_{\text{plates}} &= \frac{\hbar c}{4\pi d}\int_0^\infty k\,dk \sum_{n=-\infty}^{\infty}(k^2 + n^2\pi^2/d^2)^{1/2} \\
&= \frac{\hbar c}{8d^2}\int_0^\infty du \sum_{n=-\infty}^{\infty}(ud^2/\pi^2 + n^2)^{1/2} = \frac{\hbar c\pi^2}{8d^4}\int_0^\infty du \sum_{n=-\infty}^{\infty}(u + n^2)^{1/2} \\
&= \frac{\hbar c\pi^2}{8d^4}\sum_{n=-\infty}^{\infty}\int_{n^2}^\infty du\,\sqrt{u} = \frac{\hbar c\pi^2}{4d^4}\sum_{n=-\infty}^{\infty}\int_{|n|}^\infty d\omega\,\omega^2.
\end{aligned}$$

The expression for $\mathscr{E}_0$ is the same with the sum over n replaced by an integration over a continuous variable n.

d) All terms at $n = \infty$ vanish because of the cutoff function, and at $n = 0$ all derivatives of $\mathscr{F}$ vanish except $\mathscr{F}'''(0) = -2$.

e) The d-dependent energy $E(d)$ per unit area of the plates is $-\frac{\hbar c\pi^2}{720d^3}$, and the corresponding force on the plates is $F(d) = E'(d) = \frac{\hbar c\pi^2}{240d^4}$. The energy decreases with decreasing separation of the plates, which means that this is an attractive force.