5 Semiclassics

5.1 Triangular potential well: variational and semiclassical approximations

a) the trial function bxe^{-ax} for $x > 0$ (vanishing at $x = 0$ because of the infinite potential wall) gives eigenvalue 2.4764 in units of $(mg^2\hbar^2/2)^{1/3}$, less than 10% above the exact answer $2.33811\,(mg^2\hbar^2/2)^{1/3}$.

b) phase shift $-\pi$ (or $+\pi$, it's equivalent) at the hard wall; phase shift $-\pi/2$ at the smooth turning point, so $\gamma = -3\pi/2$

c) $p^2/2m + mgz = E \Rightarrow p = \pm\sqrt{2m(E-mgz)}$.

$$\frac{2}{\hbar}\int_0^{E/mg}\sqrt{2m(E-mgz)}\,dz - 3\pi/2 = 2\pi n \Rightarrow \frac{2}{3m^2 g}(2mE)^{3/2} = 2\pi\hbar(n+3/4)$$

so $E_n = (2m)^{-1}[3\pi\hbar m^2 g(n+3/4)]^{2/3}$.

d) $E_0 = (9\pi/8)^{2/3}[mg^2\hbar^2/2]^{1/3} = 2.320\,[mg^2\hbar^2/2]^{1/3}$; exact coefficient: 2.338, variational estimate gave 2.476.

5.2 Landau levels

a) $\boldsymbol{A} = (0, Bx, 0)$ then $\nabla \times \boldsymbol{A} = B\hat{z}$.

b) cyclotron orbit centered at the origin ($X = 0$): $p_x = \sqrt{2mE}\sin\phi$, $x = (\sqrt{2mE}/eB)\cos\phi$; two smooth turning points gives total phase shift $\gamma = -\pi$

$$\frac{2mE}{\hbar eB}\int_0^{2\pi}\sin^2\phi\,d\phi - \pi = 2\pi n \Rightarrow \frac{2\pi mE}{\hbar eB} = 2\pi(n+1/2),$$

so $E_n = (n+1/2)\hbar\omega_c$.

c) since $p_x = eBy$ the integral $\oint p_x dx = eB\oint y dx = e\Phi$ equals the flux through the cyclotron orbit, so $\Phi_n = (h/e)(n+1/2)$.

d) The quantized area is now the area enclosed between the wall and one arc of the skipping orbit; there is one hard-wall turning point and one smooth turning point, so the phase shift

γ is $-3\pi/2$ instead of $-\pi$, hence the offset is 3/4 instead of 1/2. When the cyclotron orbit center X approaches the wall, the area becomes smaller if we keep E the same, so to enclose the same flux E has to increase.

e) $p_y = mv_y + eBx = -\sqrt{2mE}\cos\phi + eBX + \sqrt{2mE}\cos\phi = eBX$; Hellmann-Feynman: $\langle v_y \rangle = dE(p_y)/dp_y$; this is negative, and indeed the skipping orbits move in the negative y-direction; the velocity goes to zero as the orbit moves further away from the wall.

f) All states at the wall move in the same direction, so the reflection probability is zero, and hence by conservation of probability the transmission probability should be one.

5.3 Resonant tunneling

a) transmitted after n round trips between the two barriers (so $2n$ reflections)

b)

$$T = \left| \sum_{n=0}^{\infty} \Gamma e^{ikL} [(1-\Gamma) e^{2ikL}]^n \right|^2 = \Gamma^2 [1-(1-\Gamma)e^{2ikL}]^{-1} [1-(1-\Gamma)e^{-2ikL}]^{-1}$$

c) If the boundaries would be impenetrable hard walls, the Bohr-Sommerfeld quantization condition would be $2kL - 2\pi = 2\pi n$, so kL is an integer multiple of π. Resonant tunneling occurs when the energy of the incident electron is aligned with a "quasi-bound" state between the two barriers.