6 Time-dependent quantum systems

6.1 Adiabatic and sudden approximations

a) initially, $E_i = (\hbar\pi/L)^2/2m$, $\psi_i = (2/L)^{1/2}\sin(\pi x/L)$;
finally, $E_f = (\hbar\pi/2L)^2/2m$, $\psi_f = (1/L)^{1/2}\sin(\pi x/2L)$; amount of work done is $E_f - E_i = -\frac{3}{4}(\hbar\pi/L)^2/2m$.

b) energy separation between ground state and first excited state is $\delta E(t) = (3/2m)[\hbar\pi/L(t)]^2$ and we need $\hbar|d\delta E/dt| \ll (\delta E)^2$, so $dL/dt \ll \hbar/mL$; so the barrier will have to slow down as L becomes larger

c) $|\langle\psi_i|\psi_f\rangle|^2 = \frac{32}{9\pi^2} = 0.36$

d) when the barrier is moved inwards ψ_i is no longer contained within the allowed region.

6.2 Berry phase in graphene

a) $H^2 = v^2(p_x^2 + p_y^2)$ times the unit matrix, so the eigenvalues $E = \pm v\sqrt{p_x^2 + p_y^2}$; this is the same energy-momentum relation as (massless) photons, with $v \mapsto c = 3\cdot 10^8$ m/s.

b) note that $e^{i\phi} = (p_x + ip_y)(p_x^2 + p_y^2)^{-1/2}$,

$$H\psi = \frac{v}{\sqrt{2}} e^{i\boldsymbol{p}\cdot\boldsymbol{r}/\hbar} \frac{p_x + ip_y}{\sqrt{p_x^2 + p_y^2}} \begin{pmatrix} \pm(p_x - ip_y) \\ (p_x^2 + p_y^2)^{1/2} \end{pmatrix}$$

take $E = \pm v\sqrt{p_x^2 + p_y^2}$, then $H\psi = E\psi$; so + sign for $E > 0$ (conduction band) and − sign for $E < 0$ (valence band).

c)

$$\gamma_{\rm B} = \frac{i}{2}\oint dt \begin{pmatrix} 1 & \pm e^{-i\phi} \end{pmatrix} e^{-i\boldsymbol{p}\cdot\boldsymbol{r}} \frac{d}{dt} \begin{pmatrix} 1 \\ \pm e^{i\phi} \end{pmatrix} e^{i\boldsymbol{p}\cdot\boldsymbol{r}} = \frac{i}{2}\oint dt \left(i\frac{d}{dt}\phi + 2i\frac{d}{dt}\boldsymbol{p}\cdot\boldsymbol{r} \right) = -\pi = \pi \bmod 2\pi.$$

d) The two turning points contribute a phase shift $\gamma = -\pi$, the Berry phase from the rotating pseudospin contributes π, so the total phase shift vanishes.

e)

$$\pi l_c^2 B = \frac{\pi p_{\rm mech}^2}{Be^2} = \frac{\pi E^2}{Be^2 v^2} = nh/e \Rightarrow E_n = \pm v\sqrt{2nB\hbar e}$$

∗) the equation of motion for the Lorentz force $\boldsymbol{F} = e\boldsymbol{v}\times\boldsymbol{B}$ *perpendicular* to the mechanical momentum $\boldsymbol{P}$ is

$$\frac{d}{dt}\boldsymbol{v} = \frac{ev}{P}\boldsymbol{v}\times\boldsymbol{B},\;\; \frac{d}{dt}\boldsymbol{P} = e\boldsymbol{v}\times\boldsymbol{B}$$

both $v = |\boldsymbol{v}|$ and $P = |\boldsymbol{P}|$ are constant; these are the same equations of motion as those of a normal electron, so the same cyclotron motion. If the force would be *parallel* to the momentum then we would have had $d\boldsymbol{v}/dt = 0$, which gives rise to an entirely different motion in an electric field (see next exercise).

6.3 Klein tunneling

a) An energy boost $E \mapsto E + E_0$ is equivalent to a shift in coordinate $x \mapsto x + E_0/F$, so it has no effect on the transmission probability from $x = -\infty$ to $x = +\infty$.

b) $x \mapsto i\hbar\partial/\partial p_x$

c) eigenvalues of H are $E_\pm = \pm(v/F)\sqrt{t^2 + p_y^2}$; gap of $\delta E = 2|p_y|v/F$ at $t = 0$; the electron starts at $x = -\infty$ where $V(x) \to \infty$, so it starts in E_+; transmission means that it ends up at $x = \infty$ where $V(x) \to -\infty$, so it ends up in E_-; the transition from E_+ to E_- is a nonadiabatic Zener transition.

d)

$$T = e^{-2\pi\Gamma},\;\; \Gamma = \frac{(\delta E/2)^2}{\left|\hbar\frac{\partial}{\partial t}(E_+ - E_-)\right|_{t=\pm\infty}} = (p_y v/F)^2(2\hbar v/F)^{-1} = \tfrac{1}{2}p_y^2 v/\hbar F$$