7 Path integrals

7.1 Lagrangian

a) $L = \frac{1}{2}m\dot{q}^2 - V$, $\Rightarrow p = \partial L/\partial\dot{q} = m\dot{q}$, $\Rightarrow \dot{q}p - L = p^2/2m + V = H$.

b) Hamilton: $\dot{p} = -\partial V/\partial q$, $\dot{q} = p/m$; Euler-Lagrange:

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{q}} - \frac{\partial L}{\partial q} = m\frac{d}{dt}\dot{q} + \frac{\partial V}{\partial q} = \dot{p} + \frac{\partial V}{\partial q} = 0.$$

c) expansion to first order in δq, partial integration, then substitute the equations of motion.

7.2 Quantum propagator

a)

$$\begin{aligned} &i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = H|\psi(0)\rangle \Rightarrow |\psi(t)\rangle = e^{-iHt/\hbar}|\psi(0)\rangle \\ &\Rightarrow \langle q|\psi(t)\rangle = \langle q|e^{-iHt/\hbar}|\psi(0)\rangle = \int dq'\,\langle q|e^{-iHt/\hbar}|q'\rangle\langle q'|\psi(0)\rangle. \end{aligned} \tag{1}$$

b)

$$G(q,q',t) = \langle q|e^{-iHt/\hbar}|q'\rangle = \sum_n \langle q|e^{-iHt/\hbar}|\psi_n\rangle\langle\psi_n||q'\rangle = \sum_n e^{-iE_nt/\hbar}\langle q|\psi_n\rangle\langle q'|\psi_n\rangle^*.$$

c)

$$G(q,q',t) = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} dp\, e^{-ip^2t/2m\hbar}e^{ip(q-q')/\hbar} = \sqrt{\frac{m}{2\pi i\hbar t}}\exp\left(\frac{im(q-q')^2}{2\hbar t}\right).$$

7.3 Feynman's formula

a) factorize the exponent (making errors of higher than first order in δt), insert the resolution of the identity for p, p', q', finally use the integral formula from exercise 2c

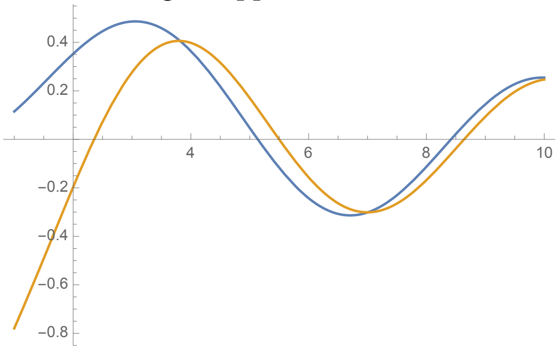
b) to prove the propagator identity, use $\int dq|q\rangle\langle q| = \hat{1}$; substituting the infinitesimal-time propagator, and retaining only the exponential factor, gives $\int dq_1\int dq_2\cdots\int dq_N\, e^{iS[q(t)]}$, with $q(t)$ the path that goes from q to q_N to q_{N-1},..., to q_2 to q_1 to q', in time steps of $deltat$. The multiple integral in the limit $N\to\infty$ becomes a path integral.

7.4 Stationary phase approximation

a)

$$\int_0^1 e^{in\pi t - ix\sin\pi t}\,dt \approx \int_0^1 e^{in\pi/2 - ix + ix(\pi^2/2)(t-1/2)^2}\,dt \approx \int_{-\infty}^{\infty} e^{in\pi/2 - ix + ix(\pi^2/2)y^2}\,dy$$
$$= \sqrt{\frac{2i}{\pi x}}e^{in\pi/2 - ix} = \sqrt{\frac{2}{\pi x}}e^{in\pi/2 + i\pi/4 - ix}. \tag{2}$$

The plot compares $J_2(x)$ and the large-x approximation.



b) $\int_{q_0}^{q} p(q')dq' - ET = \int_0^T p\dot{q}'dt - ET = \int_0^T (p\dot{q}' - H)\,dt = \int_0^T L(q',\dot{q}')dt = S[q_{\text{clas}}(t)]$

c) The action is extremal along the classical path, so the stationary phase approximation expands around that path.