1 Random Hamiltonians

1.1 Derivation of Wigner's surmise

At a conference on "Neutron Physics by Time-of-Flight", held at the Oak Ridge National Laboratory in 1957, the question was asked what the distribution might be of the spacings s of the energy levels in a nucleus (with average spacing Δ). E.P. Wigner, who was in the audience, walked up to the blackboard and guessed (= surmised) an answer,

$$P(x) = \tfrac{1}{2}\pi x \exp(-\tfrac{1}{4}\pi x^2), \;\; x \equiv s/\Delta. \tag{1}$$

Wigner's surmise turned out to be a remarkably good description of the level repulsion observed in neutron scattering.

Equation (1) is exact for an ensemble of 2×2 real symmetric matrices H with a Gaussian distribution of the independent matrix elements:

$$P(H_{11}, H_{12}, H_{22}) \propto \exp[-c(H_{11}^2 + H_{22}^2 + 2H_{12}^2)], \tag{2}$$

where c is an arbitrary constant.

Problem: Derive Eq. (1) from Eq. (2).

Help: Write H in the eigenvector-eigenvalue representation

$$\begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \tag{3}$$

and compute the Jacobian

$$J(E_1, E_2, \theta) = \frac{\partial(H_{11}, H_{12}, H_{22})}{\partial(E_1, E_2, \theta)}. \tag{4}$$

Then obtain the distribution of E_1, E_2, θ from Eq. (2) by identifying

$$P(H_{11}, H_{12}, H_{22})|J(E_1, E_2, \theta)| = P(E_1, E_2, \theta). \tag{5}$$

Finally, transform from E_1, E_2 to $\varepsilon = E_1 - E_2$, $\mathcal{E} = \frac{1}{2}(E_1 + E_2)$ and compute $P(\varepsilon)$ by integrating over $\mathcal{E}$ and θ.

1.2 Derivation of the Porter-Thomas distribution

Wigner's surmise describes the statistics of the *spacing* of the scattering resonances. The Porter-Thomas distribution describes the statistics of the *strength* of the resonances. The strength γ of the i-th resonance, in the case of a weak coupling to the j-th scattering channel, is proportional to the absolute value squared, $\gamma \propto |U_{ij}|^2$, of the unitary matrix U that diagonalizes the Hamiltonian H. Let us assume broken time-reversal symmetry, so H belongs to the Gaussian unitary ensemble of random-matrix theory.

Problem: Show that the distribution of γ in the limit $N \to \infty$ has the exponential form

$$P(\gamma) \propto \exp(-\gamma/\bar{\gamma})\theta(\gamma) \tag{6}$$

where $\bar{\gamma}$ is the mean of γ and $\theta(\gamma) = 1$ if $\gamma > 0$ and 0 if $\gamma < 0$.

Help: Consider first the marginal distribution of the elements in a row of U. Explain why

$$P(U_{11}, U_{12}, \ldots U_{1N}) \propto \delta\left(1 - \sum_{n=1}^{N} |U_{1n}|^2\right). \tag{7}$$

Integrate out $N - k$ elements in the row, to arrive at the distribution

$$P(U_{11}, U_{12}, \ldots U_{1k}) \propto \left(1 - \sum_{n=1}^{k} |U_{1n}|^2\right)^{N-k-1} \theta\left(1 - \sum_{n=1}^{k} |U_{1n}|^2\right). \tag{8}$$

Show that the individual matrix elements of U have a Gaussian distribution in the large-N limit.

Extra question: In the presence of time-reversal symmetry, H belongs to the Gaussian orthogonal ensemble. The matrix U is then orthogonal, with real matrix elements. One can show, in a similar way as above, that the distribution of the matrix elements of U is a Gaussian for $N \to \infty$. Derive the Porter-Thomas distribution

$$P(\gamma) \propto \gamma^{-1/2} \exp(-\gamma/2\bar{\gamma})\theta(\gamma). \tag{9}$$

C.E. Porter and R.G. Thomas were experimentalists. Here's the comment of Wigner on their discovery:

All of us theoreticians should feel a little embarassed. We know the theoretical interpretation of the reduced width γ: it is the value of a wave function at the boundary, and we should have been able to guess what the distribution of such a quantity is. However, none of us were courageous enough to do that... Perhaps I am now too courageous when I try to guess the distribution of the distances between successive levels.

And then Wigner went on to propose his "surmise".

1.3 Eigenvalue distribution in the Wigner-Dyson ensembles

The Wigner-Dyson ensembles of random Hamiltonian matrices H are characterized by a probability distribution $P(H) = \prod_n f(E_n)$ which depends only on a single-parameter function f of the eigenvalues E_n of H. (One typically takes a Gaussian for f, but that is not essential.) We seek the resulting distribution $P(\{E_n\})$ of the eigenvalues. For that purpose we need the Jacobian J from matrix elements to eigenvalues and eigenvectors. All correlations between the eigenvalues are due to this Jacobian.

We consider first the GOE, where H is an $N\times N$ real symmetric matrix. The volume element is

$$d\mu(H)=\prod_{i\leqslant j} dH_{ij}, \tag{10}$$

in terms of the independent matrix elements. We transform to $d\mu(H)=J\prod_i dp_i\prod_n dE_n$, where the p_i's are the degrees of freedom of the eigenvectors and J is the Jacobian of the transformation. From this expression the probability distribution of the eigenvalues follows upon integration over the p_i's,

$$P(\{E_n\})=\prod_n f(E_n)\int J\prod_i dp_i. \tag{11}$$

The Jacobian can be found by expressing the squared line element $(ds)^2=\mathrm{Tr}\, dH dH^\dagger$ in terms of the infinitesimals dE_n and dp_i (collectively denoted as dx_μ):

$$(ds)^2=\sum_{\mu,\nu} g_{\mu\nu}dx_\mu dx_\nu. \tag{12}$$

The determinant of the metric tensor g then gives the Jacobian,

$$J=|\mathrm{Det}\, g|^{1/2}. \tag{13}$$

Problem: Derive the Jacobian $J=\prod_{i<j}|E_i-E_j|$, which implies the GOE distribution

$$P(\{E_n\})\propto\prod_n f(E_n)\prod_{i<j}|E_i-E_j|^\beta,\ \ \beta=1. \tag{14}$$

Help: Write $H=ULU^{\mathrm{T}}$, where $L=\mathrm{diag}\,(E_1,E_2,\dots E_N)$ is the diagonal matrix of eigenvalues and U is a real orthogonal matrix of eigenvectors (so $U^{-1}=U^{\mathrm{T}}$, with T denoting the transpose). Show that

$$U^{\mathrm{T}}dH\,U=\delta U\,L+(\delta U\,L)^{\mathrm{T}}+dL,\ \text{ with }\delta U\equiv U^{\mathrm{T}}dU=-dU^{\mathrm{T}}\,U\equiv-\delta U^{\mathrm{T}}, \tag{15}$$

and calculate the metric tensor from $(ds)^2=\mathrm{Tr}\,(dH)^2=\mathrm{Tr}\,(U^{\mathrm{T}}dH\,U)^2$. Work out that

$$(ds)^2=\mathrm{Tr}\,(dL)^2+\mathrm{Tr}\,(\delta U\,L-L\delta U)^2=\sum_n(dE_n)^2+\sum_{i,j}(\delta U)_{ij}^2(E_i-E_j)^2. \tag{16}$$

Extra question: Consider what would change when H is complex Hermitian, to obtain the GUE distribution ($\beta=2$).

2 Random scattering matrices

2.1 Averages over the unitary group

An $N \times N$ matrix U is uniformly distributed in the unitary group $\mathscr{U}(N)$ if the averages satisfy $\langle f(U)\rangle = \langle f(U_0 U)\rangle = \langle f(U U_0)\rangle$, for any given matrix $U_0 \in \mathscr{U}(N)$ and function $f(U)$.

Problem: Prove that the average of a polynomial function

$$f(U) = U_{\alpha_1 a_1} U_{\alpha_2 a_2} \cdots U_{\alpha_p a_p} U^*_{\beta_1 b_1} U^*_{\beta_2 b_2} \cdots U^*_{\beta_q b_q} \tag{17}$$

is zero unless $p = q$ *and* the sets $\{\alpha_n\} = \{\beta_n\}$ of left indices coincide *and* the sets $\{a_n\} = \{b_n\}$ of right indices coincide (up to a permutation of the indices).
Derive the the expressions for $p = 1$ and 2,

$$\langle U_{\alpha a} U^*_{\beta b}\rangle = \frac{1}{N}\delta_{\alpha\beta}\delta_{ab}, \tag{18}$$

$$\begin{aligned}\langle U_{\alpha a} U_{\alpha' a'} U^*_{\beta b} U^*_{\beta' b'}\rangle &= \frac{1}{N^2-1}\big(\delta_{\alpha\beta}\delta_{ab}\delta_{\alpha'\beta'}\delta_{a'b'} + \delta_{\alpha\beta'}\delta_{ab'}\delta_{\alpha'\beta}\delta_{a'b}\big)\\ &\quad - \frac{1}{N(N^2-1)}\big(\delta_{\alpha\beta}\delta_{ab'}\delta_{\alpha'\beta'}\delta_{a'b} + \delta_{\alpha\beta'}\delta_{ab}\delta_{\alpha'\beta}\delta_{a'b'}\big).\end{aligned} \tag{19}$$

The coefficients that multiply the Kronecker delta's are known for arbitrary p (see the entry on *Weingarten function* in Wikipedia), but these are the only ones we will need.

2.2 Universal conductance fluctuations and weak localization

The scattering matrix of a quantum dot with a pair of N-mode point contacts is a $2N \times 2N$ unitary matrix S. It has $N \times N$ reflection and transmission submatrices r, r' and t, t',

$$S = \begin{pmatrix} r & t \\ t' & r' \end{pmatrix}. \tag{20}$$

The conductance is given by the Landauer formula,

$$G = G_0 \,\mathrm{Tr}\, t t^\dagger, \tag{21}$$

with $G_0 = 2e^2/h$ the conductance quantum. (The factor of two accounts for the electron spin.)

If time-reversal symmetry is broken by a magnetic field, S is uniformly distributed in the unitary group. This is the circular unitary ensemble (CUE). In the presence of time-reversal symmetry, $S = UU^{\mathrm{T}}$ is symmetric, with U uniformly distributed in the unitary group. This is the circular orthogonal ensemble (COE). These random matrix ensembles were introduced by F.J. Dyson in 1962 in a mathematical context, and in 1994 were applied to electrical conduction in quantum dots.

Problem: Calculate the first two moments of the conductance in the GUE and show that

$$\langle G\rangle = \frac{N}{2}\,G_0,\ \ \mathrm{Var}\,G = \frac{N^2}{16N^2-4}\,G_0^2. \tag{22}$$

For large N, the variance of the conductance becomes independent of the average conductance. This is an example of "Universal Conductance Fluctuations" (UCF).

Extra question: Calculate the average conductance in the GOE. (The variance is a bit too much work.) The result

$$\langle G\rangle = \frac{N^2}{2N+1}\,G_0 \tag{23}$$

is smaller than in the GUE. This effect is known as "weak localization".

2.3 Open transmission channels

The transmission eigenvalues T_n are eigenvalues of the transmission matrix product $tt^\dagger$. Their probability distribution in the CUE can be determined in the same way as the energy level distribution in the GUE, by calculating a Jacobian. The result is

$$P(\{T_n\}) \propto \prod_{i<j}(T_i-T_j)^2. \tag{24}$$

This distribution has the form of a Gibbs distribution of N classical particles on a line, with coordinates confined to $0<T_n<1$, and interacting with a logarithmic repulsion $U(T_n-T_m)=-\ln|T_n-T_m|$. The condition of mechanical equilibrium,

$$\int_0^1 dT'\,\rho(T')U(T-T') = \text{constant for } 0<T<1, \tag{25}$$

can be expected to accurately describe the density $\rho(T)=\langle\sum_n\delta(T-T_n)\rangle$ of transmission eigenvalues for $N\gg 1$.

Problem: Solve this integral equation, to obtain the bimodal density profile

$$\rho(T) = \frac{N}{\pi}\frac{1}{\sqrt{T}\sqrt{1-T}}. \tag{26}$$

Help: Change variables to $T=(1+\cos\theta)/2$ with $0<\theta<\pi$, and use the identity

$$\cos\theta-\cos\theta' = 2\sin[(\theta'-\theta)/2]\sin[(\theta'+\theta)/2] \tag{27}$$

to find that $\tilde{\rho}(\theta)\equiv\rho(T)dT/d\theta=$ constant solves the integral equation.

Extra question: Check that the average transmission probability equals 1/2, as expected, even though the eigenvalue density is minimal at that value. The peak at $T=1$ indicates the presence of "open transmission channels".

3 Localization and superconductivity

3.1 Localization

The probability distribution of the transmission eigenvalues in an N-mode wire (length L, mean free path l) evolves with increasing L according to the DMPK scaling equation. This equation takes a convenient form if we change variables, $T_n = 1/\cosh^2 x_n$. The probability distribution $P(\{x_n\}, L)$ of the socalled Lyapunov exponents x_n satisfies

$$l\frac{\partial}{\partial L}P = \frac{1}{2\gamma}\sum_{n=1}^{N}\frac{\partial}{\partial x_n}\left(\frac{\partial P}{\partial x_n} + \beta P\frac{\partial \Omega}{\partial x_n}\right), \text{ with } \gamma = \beta N + 2 - \beta, \tag{28}$$

$$\Omega = -\tfrac{1}{2}\sum_{i\neq j}\ln|\sinh^2 x_i - \sinh^2 x_j| - \frac{1}{\beta}\sum_{i=1}^{N}\ln|\sinh 2x_i|. \tag{29}$$

Question: Interpret this differential equation as a drift-diffusion equation for particles with coordinates $x_1, x_2, \ldots x_N$. Which variable plays the role of time? What is the diffusion constant? What is the interpretation of Ω?

As the length of the wire becomes longer and longer, the metallic behavior is lost and the wire becomes insulating. The x_n's have then diffused far apart, $1 \ll x_1 \ll x_2 \ll \cdots \ll x_N$, so the conductance $G = G_0\sum_n \cosh^{-2} x_n$ has become exponentially small, $G \propto e^{-L/\xi}$. The decay length ξ is the localization length.

Problem: Solve the DMPK equation in the limit $L \to \infty$ and show that the localization length is given by $\xi = \gamma l$.

Help: Notice that for widely separated x_n's we may approximate

$$\Omega \approx -\frac{2\gamma l}{\beta}\sum_{n=1}^{N}\frac{x_n}{\xi_n} + \text{constant}, \text{ with } \xi_n = \frac{\gamma l}{1 + \beta n - \beta}. \tag{30}$$

The solution then factorizes into independent Gaussian probability distributions $P(x_n) \propto \exp[-(\gamma l/2L)(x_n - L/\xi_n)^2]$ for each of the x_n's. The conductance in the localized regime thus has a *log-normal* distribution (meaning that $\ln G/G_0$ is Gaussian). The localization length can be defined as

$$\xi^{-1} = -\lim_{L\to\infty}\frac{1}{L}\left\langle \ln G/G_0 \right\rangle. \tag{31}$$

Comment 1: For $N \gg 1$ the localization length $\xi \approx \beta N l$ is doubled by application of a magnetic field.
Comment 2: For $N \gg 1$ the Lyapunov exponents are, on average, equally spaced by L/Nl in the localized regime. This persists in the metallic regime, for $L < \xi$, in the sense that the Lyapunov exponents have a uniform density $\rho(x) = Nl/L$ for $0 < x < L/l$.

3.2 Superconductivity

The Hamiltonian of a quantum dot connected to a superconductor by an N-mode point contact is

$$H = \begin{pmatrix} H_0 & \Delta \\ \Delta & -H_0 \end{pmatrix} = H_0 \otimes \sigma_z + \Delta \otimes \sigma_x, \tag{32}$$

with H_0 an $M \times M$ real symmetric matrix in the GOE and Δ an $M \times M$ matrix of rank N. The matrix elements Δ_{mn} are all zero, except $\Delta_{nn} = M\delta$ if $1 \leqslant n \leqslant N$. The matrix H_0 has the GOE distribution (with mean level spacing δ),

$$P(H_0) \propto \exp\left(-\tfrac{1}{4} M \lambda^{-2} \operatorname{Tr} H_0^2\right), \;\; \lambda = M\delta/\pi. \tag{33}$$

We seek the average of the density of states $\rho(E)$ of H. For that purpose we need the average $\langle \cdots \rangle$ of the Green's function

$$\mathcal{G}(z) = (z - H)^{-1} = \begin{pmatrix} \mathcal{G}_{11}(z) & \mathcal{G}_{12}(z) \\ \mathcal{G}_{21}(z) & \mathcal{G}_{22}(z) \end{pmatrix}, \;\; \rho(E) = -\frac{1}{\pi} \operatorname{Im} \operatorname{Tr} \mathcal{G}(E + i0^+). \tag{34}$$

We factor out the nonfluctuating Green's function $\mathcal{G}_0(z) = (z - \Delta \otimes \sigma_x)^{-1}$,

$$\mathcal{G}(z) = \mathcal{G}_0(z) \textstyle\sum_{p=0}^{\infty} [(H_0 \otimes \sigma_z) \mathcal{G}_0(z)]^p. \tag{35}$$

a) Verify this identity. Gaussian averages of H_0^p consist of sums of all pairwise contractions. For $M \gg 1$ only non-intersecting contractions are kept, resulting in Pastur's equation

$$\langle \mathcal{G}(z) \rangle = \mathcal{G}_0(z) + \mathcal{G}_0(z) \langle (H_0 \otimes \sigma_z) \langle \mathcal{G}(z) \rangle (H_0 \otimes \sigma_z) \rangle \langle \mathcal{G}(z) \rangle. \tag{36}$$

b) Rearrange Pastur's equation in the form

$$\langle \mathcal{G}(z) \rangle = \left[z - \Delta \otimes \sigma_x - \Sigma(z)\right]^{-1}, \;\; \Sigma(z) = \Big\langle (H_0 \otimes \sigma_z) \langle \mathcal{G}(z) \rangle (H_0 \otimes \sigma_z) \Big\rangle. \tag{37}$$

c) Explain why, for $M \gg 1$, the self-energy $\Sigma(z)$ becomes block-diagonal,

$$\Sigma(z) = \frac{\lambda^2}{M} \begin{pmatrix} g_{11}(z) & -g_{12}(z) \\ -g_{21}(z) & g_{22}(z) \end{pmatrix}, \;\; \text{with} \;\; g_{ij} = \operatorname{Tr} \langle \mathcal{G}_{ij} \rangle. \tag{38}$$

Pastur's equation then gives

$$\begin{pmatrix} g_{11}(z) & g_{12}(z) \\ g_{21}(z) & g_{22}(z) \end{pmatrix} = \sum_{n=1}^{M} \begin{pmatrix} z - (\lambda^2/M) g_{11}(z) & (\lambda^2/M) g_{12}(z) - \Delta_{nn} \\ (\lambda^2/M) g_{21}(z) - \Delta_{nn} & z - (\lambda^2/M) g_{22}(z) \end{pmatrix}^{-1}. \tag{39}$$

As a simple check, we set $\Delta \equiv 0$. Then $g_{12} = g_{21} \equiv 0$, $g_{22} = g_{11} \equiv g$ and $g(z)$ is given by

$$g(z) = \left[z/M - (\delta/\pi)^2 g(z)\right]^{-1}. \tag{40}$$

(A unique solution follows from the requirement $g \to M/z$ for $z \to \infty$.)
d) Calculate the average density of states and obtain Wigner's semicircle law for the GOE,

$$\langle \rho(E) \rangle = \delta^{-1} \sqrt{1 - (E/E_c)^2}, \;\; |E| < E_c \equiv 2M\delta/\pi. \tag{41}$$

The solution of Pastur's equation for $\Delta \neq 0$ is more complicated. (We won't do it here.) The result is that a gap opens up in the density of states: $\langle \rho(E) \rangle = 0$ for $E < E_{\text{gap}} = \gamma^{5/2} N\delta/\pi$, with $\gamma = \frac{1}{2}(\sqrt{5} - 1)$ the golden number.

4 Andreev reflection

4.1 Conductance

We consider a disordered normal-metal wire (length L large compared to mean free path l), which is connected at one end to a superconductor. An electron at the Fermi level, incident from the opposite end in mode m, is reflected into some other mode n, either as an electron (normal reflection) or as a hole (Andreev reflection), with probability amplitudes $r^{\rm ee}_{nm}$ and $r^{\rm he}_{nm}$, respectively. The $N \times N$ matrices $r^{\rm ee}$ and $r^{\rm he}$ are given by

$$r^{\rm ee} = r - t r'^{*} (1 + r' r'^{*})^{-1} t', \;\; r^{\rm he} = -{\rm i} t^{*} (1 + r' r'^{*})^{-1} t'. \tag{42}$$

The scattering matrix S of the normal metal has the polar decomposition

$$S = \begin{pmatrix} r & t \\ t' & r' \end{pmatrix} = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} \begin{pmatrix} -\sqrt{\mathscr{R}} & \sqrt{\mathscr{T}} \\ \sqrt{\mathscr{T}} & \sqrt{\mathscr{R}} \end{pmatrix} \begin{pmatrix} u' & 0 \\ 0 & v' \end{pmatrix},$$

where u, v, u', v' are $N \times N$ unitary matrices and $\mathscr{T} = 1 - \mathscr{R} = {\rm diag}\,(T_1, T_2, \ldots T_N)$ is a diagonal matrix of transmission eigenvalues.

The conductance $G_{\rm NS}$ of the normal-superconductor junction is given by the Blonder-Tinkham-Klapwijk formula,

$$G_{\rm NS} = G_0 \, {\rm Tr}\,(1 - r_{ee} r_{ee}^{\dagger} + r_{he} r_{he}^{\dagger}). \tag{43}$$

We assume zero magnetic field. Time-reversal symmetry then requires that S is a symmetric matrix, hence $u' = u^{\rm T}$, $v' = v^{\rm T}$.

Problem: Show that $G_{\rm NS}$ is given in terms of the transmission eigenvalues by

$$G_{\rm NS} = G_0 \sum_n \frac{2 T_n^2}{(2 - T_n)^2}. \tag{44}$$

This formula looks quite different from the Landauer formula for the normal-state conductance: $G_{\rm N} = G_0 \sum_n T_n$. Try the substitution $T_n = 1/\cosh^2 x_n$ and see if you can explain the resulting similarity of the two formulas in physical terms.

You might expect the superconductor to increase the conductance of the disordered wire. To see how big this effect is, calculate the average of $G_{\rm NS}$ and $G_{\rm N}$, using that the Lyapunov exponents x_n have a uniform density. *Surprise!*

4.2 Angular dependence

The equality of $\langle G_{\rm NS} \rangle$ and $\langle G_{\rm N} \rangle$ hides a secret, which reveals itself if we consider the angular distribution of the reflected electrons. This can be calculated by using that the unitary matrices in the polar decomposition of S are independently and uniformly distributed in the unitary group. We again assume zero magnetic field (so $u' = u^{\mathrm{T}}$, $v' = v^{\mathrm{T}}$).

Problem: By averaging over the unitary matrices, derive that

$$\langle |r^{\mathrm{ee}}_{nm}|^2\rangle = \frac{\delta_{nm}+1}{N^2+N}\left(N - \left\langle \sum_k \tau_k^2 \right\rangle\right), \tag{45a}$$

$$\langle |r^{\mathrm{he}}_{nm}|^2\rangle = \frac{\delta_{nm}+1}{N^2+N}\left\langle \sum_k \tau_k^2 \right\rangle + \frac{N\delta_{nm}-1}{N^3-N}\left\langle \sum_{k\neq k'} \tau_k\tau_{k'} \right\rangle, \tag{45b}$$

with the abbreviation $\tau_k \equiv T_k(2-T_k)^{-1}$.

In the large-N limit ($N \gg L/l \gg 1$) we may factorize $\langle \sum_{k\neq k'}\tau_k\tau_{k'}\rangle$ into $\langle \sum_k \tau_k\rangle^2$, which can be evaluated using the uniform density of Lyapunov exponents in a disordered wire:

$$\left\langle \sum_k f(T_k) \right\rangle = (Nl/L)\int_0^\infty dx\, f(1/\cosh^2 x). \tag{46}$$

The result for normal reflection is

$$\langle |r^{\mathrm{ee}}_{nm}|^2\rangle = (1+\delta_{nm})N^{-1}(1-\tfrac{1}{2}l/L). \tag{47}$$

Off-diagonal ($n \neq m$) and diagonal ($n = m$) reflection differ by precisely a factor of two, an effect known as *coherent backscattering*. In contrast, for Andreev reflection we find

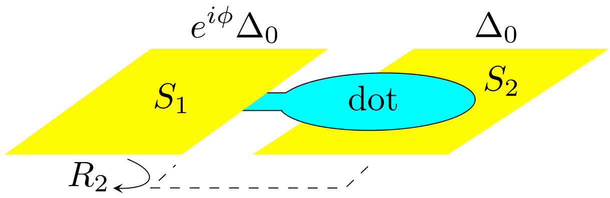
$$\langle |r^{\mathrm{he}}_{nm}|^2\rangle = \tfrac{1}{2}l/NL \;\; (n\neq m),\;\; \langle |r^{\mathrm{he}}_{nn}|^2\rangle = (\pi l/4L)^2. \tag{48}$$

Off-diagonal and diagonal reflection now differ by an order of magnitude $Nl/L \gg 1$. This *giant backscattering* peak can be detected by measuring the conductance via a point contact.

Extra question: Show that if time-reversal symmetry is broken by a magnetic field, the difference between the off-diagonal and diagonal reflection amplitudes disappears.

5 Topological superconductivity

A superconductor with broken time-reversal and broken spin-rotation symmetry is called "topological", because it can exhibit a phase transition where a topological quantum number changes. In a confined geometry (quantum dot), the topological quantum number is the parity of the total number of electrons in the ground state. A switch between an even and odd number of electrons in the ground state is signaled by a level crossing at the Fermi level ($E = 0$). In this exercise we will study the statistics of these fermion parity switches in the context of RMT.



We consider a quantum dot coupled to superconductors S_1 and S_2. We choose a gauge where the order parameter Δ_0 in S_2 is real, while S_1 is phase biased at $e^{i\phi}\Delta_0$. The excitation spectrum of this Josephson junction is discrete for $|E| < \Delta_0$ and $\pm E$ symmetric because of electron-hole symmetry. As ϕ is advanced by 2π, pairs of excitation energies $\pm E_n(\phi)$ may cross. One might thing that such level crossings are prevented by level repulsion, but as we will see that is not the case.

Electrons and holes (e, h) at the Fermi level propagate through the point contact between S_1 and the quantum dot in one of the $N = 2M$ modes. (The factor of two accounts for the $\uparrow,\downarrow$ spin degree of freedom.) Left-moving quasiparticles are Andreev reflected by S_1 and right-moving quasiparticles are reflected by the quantum dot coupled to S_2. The vector $\Psi = (\Psi_{e\uparrow}, \Psi_{e\downarrow}, \Psi_{h\uparrow}, \Psi_{h\downarrow})$ of wave amplitudes is transformed as $\Psi \mapsto R_2 R_1 \Psi$, by multiplication with the reflection matrices

$$R_1(\phi) = \begin{pmatrix} 0 & e^{-i\phi}\Lambda \\ e^{i\phi}\Lambda^{\mathrm{T}} & 0 \end{pmatrix}, \quad \Lambda = \bigoplus_{m=1}^{M} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \tag{49a}$$

$$R_2 = \begin{pmatrix} r_{ee} & r_{eh} \\ r_{he} & r_{hh} \end{pmatrix}, \quad r_{hh} = r_{ee}^*, \quad r_{eh} = r_{he}^*. \tag{49b}$$

These are $2N \times 2N$ unitary matrices, with four $N \times N$ subblocks related by electron-hole symmetry. The condition for a level crossing at phase ϕ is that Ψ is an eigenstate of $R_2 R_1(\phi)$ with unit eigenvalue, so

$$\mathrm{Det}\,[1 - R_2 R_1(\phi)] = 0. \tag{50}$$

We seek to rewrite this as an eigenvalue equation for some real matrix $\mathcal{M}$. For that purpose we change variables from phase $\phi \in (-\pi, \pi)$ to quasienergy $\varepsilon = \tan(\phi/2) \in (-\infty, \infty)$. Eq. (50) then takes the form

$$\mathrm{Det}\,[1 - U - i\varepsilon(1 + U)\tau_z] = 0, \quad \tau_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \text{ with } U = R_2 R_1(0). \tag{51}$$

a) Verify this identity.

The complex unitary matrix U becomes a real orthogonal matrix O upon a change of basis,

$$O = \Omega^\dagger U \Omega, \quad \Omega = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}. \tag{52}$$

Note that $\mathrm{Det}\, O = \mathrm{Det}\, U = 1$, so $O \in \mathrm{SO}(2N)$ is special orthogonal. Since $\Omega^\dagger \tau_z \Omega = -\tau_y$, the level crossing condition becomes

$$\mathrm{Det}\,[1 - O + \varepsilon(1+O)J] = 0,\;\; J = i\tau_y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{53}$$

b) Can you also verify this identity?

For chaotic scattering O is uniformly distributed with the Haar measure of $\mathrm{SO}(2N)$. This is the circular real ensemble (CRE) of random-matrix theory in symmetry class D. The special orthogonal matrix O can be represented by an antisymmetric real matrix $A = -A^{\mathrm{T}}$, through the Cayley transform

$$O = (1-A)(1+A)^{-1}. \tag{54}$$

c) The Cayley transform does not exist if $\mathrm{Det}\, O = -1$. *Do you have an idea what this corresponds to physically?*

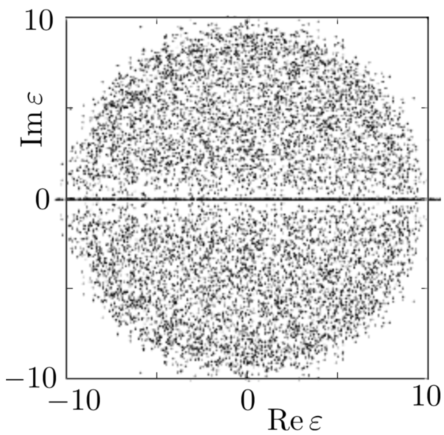
Substitution of Eq. (54) into Eq. (53) gives the level crossing condition as an eigenvalue equation,

$$\mathrm{Det}\,(\mathscr{M} - \varepsilon) = 0,\;\; \mathscr{M} = AJ = (1-O)(1+O)^{-1}J. \tag{55}$$

The matrix $\mathscr{M}$ is real but not symmetric: $\mathscr{M}^{\mathrm{T}} = -J\mathscr{M}J$. This is the definition of a *skew-Hamiltonian* matrix. There are N distinct eigenvalues, each with multiplicity two. The N_X distinct real eigenvalues ε_n identify the level crossings at $\phi_n = 2\arctan\varepsilon_n$.

d) Why does each eigenvalue of $\mathscr{M}$ *have multiplicity two?*

We have thus transformed the level crossing problem into a classic problem of RMT: *How many eigenvalues of a real matrix are real?* One might have guessed that an eigenvalue is exactly real with vanishing probability, since the real axis has measure zero in the complex plane. Instead, the eigenvalues of real non-Hermitian matrices accumulate on the real axis (see figure).



e) Can you explain this accumulation?

The number of eigenvalues on the real axis scales as $\sqrt{4N/\pi}$, so this is the expected number of topological phase transitions when ϕ is advanced by 2π.

f) Can you come up with a qualitative argument for the $\sqrt{N}$ *scaling?*