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CONTENTS

DUALS

by

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Notations.

We denote by $\mathbb{Q}_p$ the field of p-adic numbers, $\mathbb{Z}_p$ its ring of integers, $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ its residue field. The absolute value $|x|$ and valuation $v_p(x)$ are normalised by $|p| = 1/p$ and $v_p(p) = 1$. Then $\mathbb{C}_p$ is the completion of the algebraic closure of $\mathbb{Q}_p$, provided with the only absolute value and valuation inducing those given on $\mathbb{Q}_p$, $\mathcal{O}_p$ and $\mathfrak{M}_p$ denote the valuation ring and valuation ideal of $\mathbb{C}_p$. In general, K will denote a p-adic field, that is $\mathbb{Q}_p \subseteq K \subseteq \mathbb{C}_p$, $\mathcal{O}_K$, $\mathfrak{M}_K$ and k (or $\mathcal{O}$, $\mathfrak{M}$ and k) are the valuation ring, valuation ideal and residue field of K.

The open disk (resp. closed disk) of radius r containing a, $\{x \in K \mid |x-a| < r\}$ (resp. $\{x \in K \mid |x-a| \leq r\}$) is denoted by $D(a,r^-)$ (resp. $D(a,r^+)$). If D is a disk, $A(D)$ is the ring of analytic functions on D.

Introduction.

Among the meanings which could be given to "duals" in p-adic analysis, we will choose two:

(i) dual group: let G be a profinite group, say $G = \varprojlim G_\alpha$, where the G_α are finite, and define $G^*_{(p)} = \text{Hom.cont.}(G, \mathbb{C}^*_p)$ as the group of continuous morphisms from G to the multiplicative group $\mathbb{C}^*_p$. Then $G^*_{(p)}$ is the p-adic dual group of G. When p is fixed, we call it the dual of G, and denote it by G^*.

(ii) dual space: Let E be a topological vector space on a p-adic field K, the space E' of continuous K-linear maps from E to K is the dual space of E. If $\mu \in E'$ and $x \in E$, we denote the value of μ at x by $\mu(x)$ or $(\mu|x)$. In the case when E is a normed space, E' is provided with its natural norm $||\mu|| = \sup_{||x|| \leq 1} |\mu(x)|$.

Examples.

(a) Let $G = \mathbb{Z}_p$, then the map $\theta \to \theta(1)$ from $\mathbb{Z}_p^*$ to $\mathbb{C}_p^*$ is an isomorphism of $\mathbb{Z}_p^*$ onto $1 + \mathfrak{M}_p$. If $\mathbb{Z}_p^*$ is provided with the metric it receives from $C(\mathbb{Z}_p, K)$ (with uniform convergence norm), then $\theta \to \theta(1)$ is isometric.

(b) Take $G = \hat{\mathbb{Z}} = \varprojlim \mathbb{Z}/N\mathbb{Z}$, with the obvious morphisms when $M|N$. Then, for a given p, $\hat{\mathbb{Z}} \simeq \mathbb{Z}_p \times \prod_{p' \neq p} \mathbb{Z}_{p'}$. Here, the map $\theta \to \theta(1)$ from $\hat{\mathbb{Z}}_{(p)}$ to $\mathbb{C}_p^*$ is an isometric isomorphism onto $\mathbb{T}_p = \{x \in \mathbb{C}_p \mid |x| = 1\}$.

(c) Let $c_0(K) = \{a = (a_n)_{n \in N} \mid a_n \in K, |a_n| \to 0 \text{ when } n \to \infty\}$, with the norm $||a|| = \operatorname{Max} |a_n|$. The dual space is the space $b(K) = \{b = (b_n)_{n \in N} \mid b_n \in K, \text{ and } \operatorname{Max} |b_n| < +\infty\}$ of bounded sequences, with the maximum norm.

(d) Every time when $G \simeq \mathbb{Z}_p^d \times H$, where H has a finite p-component (H is supposed to be profinite and, if $H = \varprojlim H_\alpha$, the p-component of H is the limit of the p-components of the H_α), then $G^* \simeq (1 + \mathfrak{M}_p)^d \times H^*$ where H^* is discrete. As $(1 + \mathfrak{M}_p)^d$ is a polydisk $|z_i - 1| < 1$, $i = 1, \ldots, d$ of $\mathbb{C}_p^d$, it will make sense to speak of analytic functions on G^*.

As G^* is contained in the space $C(G, \mathbb{C}_p)$ of continuous functions on G (with the uniform convergence norm), the dual space of $C(G, \mathbb{C}_p)$ is also a space of functions on the dual group G^*. More generally, if F is a subspace of $C(G, \mathbb{C}_p)$, its dual space F' is a space of functions on $G^* \cap F$. Our purpose here is to give characterizations

of such F' (dual spaces of "natural" subspaces of $C(G, \mathbb{C}_p)$, viewed as spaces of functions on (part of) G^*. For a general study of dual groups, see for instance [9]; another useful viewpoint for the description of dual spaces is developped in [3].

Most of the techniques explained here have been used without proof in [4], and were built by the authors of [4] at that time. It seems that untill now, there still do not exist exposition of this kind of techniques. In 3.2 we show a new application.

1. Measures, distributions.

To avoid the heavy notations which would be necessary to express the following notions and results in the general case of a profinite group (as in example (d) of introduction) we will give most statements in the case when $G = \mathbb{Z}_p$. It is very easy to generalize the results first to $\mathbb{Z}_p^d$ and then to $\mathbb{Z}_p^d \times H$, using tensor products to get several variables (as in [1], chap.3, for instance).

1.1 Basic example. [2].

Let $E_K = C(\mathbb{Z}_p, K)$ be the space of continuous functions on $\mathbb{Z}_p$ with values in the field K, provided with the maximum norm, its dual space E_K' is called the space of measures on $\mathbb{Z}_p$. As the space generated by the characteristic functions of clopen sets is dense in $C(\mathbb{Z}_p, K)$, one sees that to give a measure is to give an additive and bounded function on the set of characteristic functions of clopen sets. If μ is a measure, let $\mu_n = (\mu|\binom{x}{n})$, then the sequence $(\mu_n)_{n \in N}$ is bounded and the map $\mu \to (\mu_n)_{n \in N}$ is an isometric isomorphism of E_K' onto $b(K)$.

If $K = \mathbb{C}_p$, then we write $E_K = E$, in this case, $\mathbb{Z}_p^* \subseteq E$. Let θ be any p-adic character of $\mathbb{Z}_p$ (i.e. $\theta \in \mathbb{Z}_p^*$) then, uniformly with respect to $x \in \mathbb{Z}_p$ one has: $\theta(x) = \sum_{n\geq 0} (\theta(1)-1)^n \binom{x}{n}$. Hence, if $\mu \in E'$, $(\mu|\theta) = \sum_{n\geq 0} \mu_n(\theta(1)-1)^n$. This shows that <u>the space of restrictions to $\mathbb{Z}_p^*$ of measures on $\mathbb{Z}_p$ is the space of bounded analytic functions on $\mathbb{Z}_p^*$</u>.

To define a measure $\mu \in E'$, it is enough to give its values on the locally constant functions: in terms of functions on the dual group $\mathbb{Z}_p^*$ that means that a bounded analytic function is known when you know its values on the torsion subgroup of $\mathbb{Z}_p^*$.

A study of properties of these analytic functions in relation with local properties of the corresponding measures is given in [6] by D. BARSKY.

1.2 <u>Distributions</u>.

Let G be profinite, $G = \varprojlim (G_\alpha)$, with G_α finite, and let Loc.cst.(G,K) denote the space of locally constant functions on G with values in K. Any K-linear map on Loc.cst.(G,K) is called a <u>distribution</u> on G. As Loc.cst.(G,K) is dense in $\mathcal{C}(G,K)$, a distribution which is a bounded linear map can be uniquely continued to $\mathcal{C}(G,K)$ as a measure. Let Dist(G,$\mathcal{O}$) denote the $\mathcal{O}$-module of distributions on G with values in the ring $\mathcal{O}$, then one can (easily) prove (see for instance [12]):

PROPOSITION 1.2. $\mathrm{Dist}(G,\mathcal{O}) = \mathcal{O}[G] = \varprojlim \mathcal{O}[G_\alpha]$.

In this case when $G = \mathbb{Z}_p$, as $\mathcal{O}[\mathbb{Z}/p^n\mathbb{Z}] = \mathcal{O}[X]/X^{p^n}-1$, one gets $\mathcal{O}[\mathbb{Z}_p] = \mathcal{O}[[(X-1)]]$. If $\mathcal{O}$ is the valuation ring of $\mathbb{C}_p$, then $\mathrm{Dist}(\mathbb{Z}_p,\mathcal{O})$ is exactly the unit ball of E' and the distributions with values in $\mathcal{O}$ are nothing but measures with norm smaller than 1. On the contrary, distributions with values in an unbounded ring, like a p-adic field, are not measures but a strictly bigger space.

For instance the well-known "Haar measure" on $\mathbb{Z}_p$ (which is not a measure in the sense developped here), defined by $\mu(1_D) = p^{-n}$ if 1_D is the characteristic function of the disk $D(a,p^{-n})$, is a distribution on $\mathbb{Z}_p$.

1.3 <u>Tempered distributions</u>.

Let F be any subspace of E_K, containing the locally constant functions. Then the elements of the dual space of F define distributions on $\mathbb{Z}_p$. In other words, a distribution is the restriction to the torsion subgroup of $\mathbb{Z}_p^*$ of any function on $\mathbb{Z}_p^*$.

We will now look at a special subspace of $\mathcal{C}(\mathbb{Z}_p,K)$, the space $LA(\mathbb{Z}_p,K)$ of <u>locally analytic functions</u>. We recall that a function f on $\mathbb{Z}_p$ is said locally analytic if, for every $a \in \mathbb{Z}_p$ there exists a disk $D_a \ni a$ such that the restriction of f to D_a is equal to the sum of a convergent entire series in $(x-a)$. For $h \geq 0$, $LA_h(\mathbb{Z}_p,K)$ denotes the space of f whose restriction to each disk $D(a,p^{-h})$ is strictly analytic on this disk. As a direct sum of spaces of restricted series, LA_h has a natural norm. The compactness of $\mathbb{Z}_p$ shows that $LA = \bigcup_h LA_h$. The natural imbeddings $LA_h \longrightarrow LA_{h+1}$ are continuous and LA is provided with its (direct) limit topology. The space LA contains the dual group $\mathbb{Z}_p^*$.

One knows [1] that, if $f \in E$, $f(x) = \sum_{n\geq 0} a_n\binom{x}{n}$, then $f \in LA$ if and only if $\underline{\lim}\ (v(a_n)/n) > 0$. Moreover, if $w_h(f) = \mathrm{Inf}_n\ (v(a_n) - v([n/p^h]!))$, then $A_h(f) = p^{-w_h(f)}$ defines a norm on LA_h, which coincides with the natural one.

Hence, if μ is a linear form on the space of the polynomials, let $\mu_n = (\mu|\binom{x}{n})$, then μ is continuous with respect to the topology induced on the polynomials by LA if and only if, for all sequences (a_n) satisfying $\underline{\lim}\ (v(a_n)/n) > 0$, the series $\Sigma\ \mu_n a_n$ converges. That means that the dual space LA' of LA is the space of the μ such that the series $\Sigma\ \mu_n T^n$ converges for $|T| < 1$. In particular applying such μ to elements of the dual group $\mathbb{Z}_p^*$, which generate a dense subspace of LA, one gets the:

THEOREM 1.3. Let us call tempered distributions the elements of $LA'(\mathbb{Z}_p,K)$. The space of restrictions to $\mathbb{Z}_p^*$ of tempered distributions on $\mathbb{Z}_p$ is the space of analytic functions on $\mathbb{Z}_p^*$.

In the following we will make use of these notations: if $\mu \in LA'$ and $\theta \in Z_p^*$ one has $\mu(\theta) = (\mu|\theta) = \sum_{n\geq 0} \mu_n(\theta(1)-1)^n$. We denote by $\mu(X) = \sum_{n\geq 0} \mu_n(X-1)^n$ the Taylor series associated to μ as well, if useful, as the function analytic on $D(1,1^-)$ which it defines. This amounts to an identification of LA' with a ring of formal series converging in a disk of radius 1 and with the ring $A(D(1,1^-))$.

In the next section we show the correspondence between some spaces of continuous functions (lying between $C(\mathbb{Z}_p,\mathbb{C}_p)$ and $LA(\mathbb{Z}_p,\mathbb{C}_p)$) and their duals which appear as spaces of analytic functions on $\mathbb{Z}_p$, with growth condition "on the boundary".

2. Growth conditions.

2.1 Lipschitz conditions of order k, space Lip_k (see D. BARSKY [5]).

If $f \in E_K$ and $m_1 \in \mathbb{Z}_p - \{0\}$, let

$$\Delta_{m_1}(f)(x) = \frac{f(x+m_1)-f(x)}{m_1}, \quad \text{and,}$$

if $(m_1,m_2,\ldots,m_k) \in (\mathbb{Z}_p-\{0\})^k$, $\Delta_{m_1,\ldots,m_k}(f)(x) = \Delta_{m_k}(\Delta_{m_1,\ldots,m_{k-1}})(f)(x)$.

Denote by $w_k(f) = \mathrm{Inf}\ (v(\Delta_{m_1,\ldots,m_k}(f)(x))$ where the Inf is taken on the set of $(x,m_1,\ldots,m_k) \in \mathbb{Z}_p \times (\mathbb{Z}_p-\{0\})^k$.

DEFINITION. We call $Lip_k(\mathbb{Z}_p,K)$ the set of f belonging to E_K for which $w_k(f) > -\infty$. Provided with the norm

$$N_k(f) = p^{-w_k(f)}, \quad Lip_k \text{ is a Banach space.}$$

We say that the functions f belonging to Lip_k satisfy a Lipschitz condition of order k.

DEFINITION (bis). The function f is said to be k times continuously and uniformly differentiable if and only if it has a continuous k-th derivative $f^{(k)}$ and $\Delta_{m_1,\ldots,m_k}(f)(x) \to f^{(k)}(x)$, uniformly with respect to $x \in \mathbb{Z}_p$, when $(m_1,\ldots,m_k) \to 0$. The space $C_u^{(k)}$ of those functions is a closed subspace of Lip_k.

These subspaces of E_K have a nice characterization in terms of interpolating series, due to D. BARSKY [5].

THEOREM 2.1 Let $f(x) = \sum_{n\geq 0} a_n\binom{x}{n} \in E_K$, then $w_k(f) = \mathrm{Inf}_{0\leq r\leq k,\, n\geq r} (v(a_n)-L(n,r))$ where $L(n,r) = \mathrm{Max}\{(v(n_1)+\ldots+v(n_r)) \mid n_1+\ldots+n_r \leq n,\ n_i \geq 1\}$.

One can prove the following properties of $L(n,r)$:

(i) for any fixed j, $\lim_{n\to+\infty}(L(n,j)-L(n,j-1)) = +\infty$.

(ii) for any fixed j, there exists $H(j)$ such that, for all n, $|L(n,j)-jL(n,1)| \leq H(j)$.

Moreover, it is almost obvious that $1 \leq np^{-L(n,1)} \leq p$.

THEOREM 2.1 bis. Let $f(x) = \sum_{n>0} a_n \binom{x}{n} \in E_K$, then $f \in C_u^{(k)}$ if and only if $v(a_n) - L(n,k) \to +\infty$ when $n \to +\infty$.

Together with the above properties of the function $L(n,r)$, these two theorems give the following characterizations.

COROLLARY 2.1.1 Let $f(x) = \sum_{n>0} a_n \binom{x}{n} \in E_K$, then $f \in \mathrm{Lip}_k$ if and only if $M_k(f) = \mathrm{Max}(|a_0|, \mathrm{Max}_n (n^k |a_n|)) \to +\infty$, and $M_k(f)$ defines on Lip_k a norm equivalent to N_k.

COROLLARY 2.1.2 Let $f(x) = \sum_{n>0} a_n \binom{x}{n} \in E_K$, then $f \in C_u^{(k)}$ if and only if $n^k |a_n| \to 0$ as $n \to +\infty$.

Remarks. It is clear that, for $k \geq 0$, $E_K \supseteq \mathrm{Lip}_k \supseteq C_u^{(k)} \supseteq \mathrm{Lip}_{(k+1)} \supseteq LA$. Moreover, if $\mathrm{Loc.pol.}^{(k)}$ denotes the space of those functions which are locally polynomials of degree at most k, then $\mathrm{Loc.pol}^{(k)} \subseteq C_u^{(k)}$ and $\mathrm{Loc.pol.}^{(k)}$ is dense in $C_u^{(k)}$.

2.2 Growth conditions on the boundary of an open disk.

The inclusions noticed in the above remarks show that the dual spaces of Lip_k and $C_u^{(k)}$ lie between the space of all analytic functions on the dual $\mathbb{Z}_p^*$ and the space of bounded analytic functions. We will now show that those spaces are defined by growth conditions in the following sense.

We recall that, if $F \in A(D(1,1^-))$, for $0 \leq r < 1$, one defines $M(F,r) = \mathrm{Max}(|F(x)| \mid |x-1| \leq r,\ x \in \mathbb{C}_p)$. One knows that $M(F,r)$ is a continuous non-decreasing function of r. If $F(X) = \sum a_n (x-1)^n$ is the Taylor series of F at $X = 1$, $M(F,r) = \mathrm{Max}_{n\geq 0} |a_n| r^n$.

Notation. Let F and $G \in A(D(1,1^-))$, we write $F = \mathcal{O}(G)$ (resp. $F = o(G)$) whenever $M(F,r) = \mathcal{O}(M(G,r))$ (resp. $M(F,r) = o(M(G,r))$) as $r \to 1^-$.

If $F = \mathcal{O}(G)$ and $G = \mathcal{O}(F)$ we say that F and G have the same growth on the boundary of the disk $D(1,1^-)$. For instance, two bounded non-zero functions (analytic, of course) have the same growth. If P is a polynomial, $\mathrm{Log}(X)$ and $\mathrm{Log}(P(x))$ have the same growth. One must remark that these conditions do not have the same meaning as in, say, real analysis: here, if $F = \mathcal{O}(G)$, F need not be zero at the zeroes of G "tending to the boundary of the disk".

Examples. Let $F(X) = \sum_{n>0} a_n (X-1)^n$, then,

- $F = \mathcal{O}(\mathrm{Log}) \iff n^{-1}|a_n|$ bounded;
- $F = o(\mathrm{Log}) \iff n^{-1}|a_n| \to 0$;
- $F = \mathcal{O}(\mathrm{Log}^k) \iff n^{-k}|a_n|$ bounded;
- $F = o(\mathrm{Log}^k) \iff n^{-k}|a_n| \to 0$.

One can easily see that a "change of coordinate" in the disk $D(1,1^-)$, that is a transformation $X \to aX+b$ with $|a| = 1$ and $|b-1| < 1$, can change the functions $M(F,r)$ for $r < |b-1|$, but not for $r \geq |b-1|$. This shows that the notions $\mathcal{O}$ and o as defined above make sense for analytic functions on $\mathbb{Z}_p^*$ (for instance) and do not depend on the choice we have made of a coordinate on $\mathbb{Z}_p^*$ (for instance, if at $\mathbb{Z}_p$ and $|a| = 1$, $\theta \to \theta(a)$ is another coordinate on $\mathbb{Z}_p^*$, as it gives another isomorphism between $\mathbb{Z}_p^*$ and the group $D(1,1^-)$, but the notions of growth conditions for analytic functions on $\mathbb{Z}_p^*$ which one can derive from this choice do not depend on a).

Now, if one put together the corollaries 2.1.1, 2.1.2 and the examples given above, one gets the:

THEOREM 2.2 Let $\mu \in LA'$, then, for $k \geq 1$:

$\mu \in (\mathrm{Lip}_k)'$ if and only if $\mu = o(\mathrm{Log}^k)$

$\mu \in (C_u^{(k)})'$ if and only if $\mu = \mathcal{O}(\mathrm{Log}^k)$.

This result was a fundamental tool in the constructions of [4].

2.3 Operators.

In this section we show the transposed operators on the dual spaces of some natural operators on the considered spaces.

2.3.1 Derivative in $A(\mathbb{Z}_p^*)$

Let $D = X\frac{d}{dX}$ be the natural invariant derivation in $A(\mathbb{Z}_p^*)$, and let D^* be the transposed operator on LA. By definition D^* is such that, if $f \in LA$, D^*f is the function such that, for all $\mu \in LA'$, $(D\mu|f) = (\mu|D^*f)$. Then $D^*f(t) = tf(t)$.

The proof is a plain calculation: if $\mu_n = (\mu|\binom{t}{n})$, $(D\mu|\binom{t}{n}) = n\mu_n + (n+1)\mu_{(n+1)}$. So, $(D\mu|\binom{t}{n}) = (\mu|n\binom{t}{n}+(n+1)\binom{t}{n+1}) = (\mu|t\binom{t}{n})$. Hence, if $f(t) = \sum_{n>0} a_n\binom{t}{n}$, $D^*f(t) = tf(t)$.

One can as well view D as an operator on the subspaces $\mathcal{O}(\mathrm{Log}^k)$, $o(\mathrm{Log}^k)$ or $O(1)$, so that D^* is then an operator on Lip_k, $C_u^{(k)}$ or E_K.

2.3.2 Derivative in LA.

Let $d = \frac{d}{dt}$ be the derivative with respect to $t \in \mathbb{Z}_p$, this is a continuous linear operator on $LA(\mathbb{Z}_p,K)$. The transposed operator d^* on $A(\mathbb{Z}_p^*) = LA'$ is $d^*(\mu(X)) = \mathrm{Log}\, X.\mu(X)$.

The proof of this last relation is easy: let $\theta \in \mathbb{Z}_p^*$, then $d(O)(t) = \mathrm{Log}\ d(\theta) = \mathrm{Log}(\theta(1))\theta$, so for $\theta \in \mathbb{Z}_p^*$, $d^*\mu(\theta) = \mathrm{Log}(\theta(1))\mu(\theta)$.

2.3.3 Multiplying by a function on $\mathbb{Z}_p$.

Let $g \in LA$, M_g be the operator on LA "multiplication by g", and M_g^* the transposed operators on LA'. One can give an abstract characterization of this class of operators on LA'. At least, we know yet that the derivative D is one of them, as $D = M_t$. Another important case is when $g_a(t) = a^t$ with $a \in D(1,1^-)$ then $M_{g_a}^*(\mu)(X) = \mu(aX)$.

2.3.4 Convolution of tempered distributions.

The space LA' of tempered distribution has a natural structure of algebra, as a ring of functions on $\mathbb{Z}_p^*$. Another natural way to multiply elements of LA' is the convolution: let μ and $\nu \in LA'$, then, define $\mu \otimes \nu$ as a continuous linear map on $LA \hat{\otimes} LA = LA(\mathbb{Z}_p^2)$, then, by definition $\mu * \nu$ is the element of LA' such that, for $f \in LA$, $(\mu * \nu \mid f) = (\mu \otimes \nu \mid f(x+y))$. When applied to $f = \theta \in \mathbb{Z}_p^*$ this process gives $(\mu * \nu \mid \theta) = (\mu \mid \theta)(\nu \mid \theta)$ so that the convolution of distributions on $\mathbb{Z}_p$ corresponds to the usual product in the algebra $A(\mathbb{Z}_p^*)$.

Given a fixed $\mu \in LA'$, the operator "convolution with μ" has a transposed operator on LA. For instance if $\mu(X) = \mathrm{Log}\ X$ the transposed operator is d as we have seen in 2.3.2. Another ususal example is when $\mu = \delta_a$ (Dirac measure in $a \in \mathbb{Z}_p$) the transposed operator of the convolution by δ_a is the translation $f(t) \to f(t+a)$.

2.3.5 Restriction.

Let U be a clopen subset of $\mathbb{Z}_p$ and R_U the operator on E defined by $R_Uf(t) = f(t)$ if $t \in U$ and 0 if $t \notin U$. This is a special case of 2.3.3 where one multiplies by the characteristic function of U. As characteristic functions of disks of $\mathbb{Z}_p$ are linear combinations of functions g_a where a is a p^n-th root of 1 for a suitable n, the transposed operator is a linear combination of operators $\mu(X) \to \mu(aX)$. For instance, if $U = \mathbb{Z}_p - p\mathbb{Z}_p$ is the set of untis of $\mathbb{Z}_p$, then one shows easily that

$$R_U(\mu)(X) = \mu(X) - (1/p) \sum_{\xi^p=1} \mu(\xi X).$$

2.4 Moments.

Let $F \in LA'$ and write $c_n(F) = (F|t^n)$. As F is entirely defined by the sequence $b_n(F) = (F|\binom{t}{n})$, it is clear that F is also defined by the sequence $c_n(F)$. We call the sequence $(c_n(F))$: the sequence of the moments of F.

A natural "problem of the moments" is then, given a sequence (c_n), to know whether it is the sequence of the moments of some F in LA'. The following lemma gives an answer.

LEMMA 2.4 *Let $F \in LA'$ and $c_n = (F|t^n)$ be the sequence of its moments. Then if* $G(T) = \sum_{n\geq 0} c_n(T^n/n!)$, $G(T) = F(\exp T - 1)$.

The proof is almost formal. From the formal identity $\exp(XT) = \sum_{n\geq 0} (X^nT^n/n!) = \sum_{k\geq 0} \binom{X}{k}(\exp T-1)^k$, one can derive that, for any $t \in \mathbb{C}_p$ such that $v(t) > 1/p-1$, and uniformly with respect to $x \in \mathbb{Z}_p$, $\exp(xt) = \sum_{n\geq 0} (x^nt^n)/n! = \sum_{k\geq 0} \binom{x}{k}(\exp t-1)^k$. Then applying F to both sides, for a fixed t, one gets

$\sum_{n\geq 0} c_n t^n/n! = \sum_{k\geq 0} b_k(\exp t-1)^k$, which proves the lemma.

Of course, one can get as corollaries of this lemma necessary and sufficient conditions for a sequence c_n to be the sequence of the moments of an F in $(Lip_k)'$, or in $(C_u^{(k)})'$, or of a measure. In the case of a measure, that gives a slightly different description equivalent to the one given in [15] §4.2.

3. Applications.

In most applications, which are arithmetical, the arithmetical background is quite heavy and would make another paper necessary. That is the reason why we give here only an abstract of two applications for which references exist.

3.1 The p-adic L-series associated to Hecke series. Yu.I. MANIN [13] and [14], Y. AMICE and J. VELU [4].

In this example, $G = \mathbb{Z}_p^\times \times H$ where $\mathbb{Z}_p^\times$ is the group of units of $\mathbb{Z}_p$ and H is finite. Then $G^* = D(1,1^-) \times H_1$, where H_1 is finite. Let $\theta : G \to 1+q\mathbb{Z}_p$ be the natural projection, (where, as usual, $q = p$ if $p \neq 2$, $q = 4$ if $p = 2$) and let χ be a character of G, of order a power of p. Then one wants to build a L-function as a function on the dual group G^*, taking pre-assigned values $b_{\chi,k}$, on the points $\chi\theta^k$, $k = 0,\dots,w$; here χ goes through all characters of order a p-power. It is quite easy to see that the uniqueness of such a function will be insured if one asks the function to be $\mathcal{O}(\mathrm{Log}^{w+1})$, that is to belong to $(Lip_{w+1})'$. After that the existence problem is solved by interpolation techniques which are completely stranger to this paper.

3.2 Bernoulli distribution.

One can formally define $F(t,X) = \dfrac{t\,e^{tX}}{e^t-1} = \sum_{k\geq 0} B_k(X)\dfrac{t^k}{k!}$ and if f is a function on $\mathbb{Z}$, periodic with period N, one defines

$$F_f^{(N)}(t,X) = \frac{1}{N}\sum_{a=0}^{N-1} f(a)\,\frac{t\,e^{(a+x)t}}{e^{Nt}-1} = \frac{1}{N}\sum_{a=0}^{N-1} F(Nt,\frac{x+a}{N})f(a) = \sum_{k\geq 0} B_{k,f}^{(N)}(X)\frac{t^k}{k!}.$$

Now, if $x \in \mathbb{Q}$, write $x = [x] + \langle x\rangle$ with $[x] \in \mathbb{Z}$ and $\langle x\rangle \in [0,1[$, then one can prove the following relation (see for instance [12]):

$$N^{k-1}\sum_{a \bmod N} B_k(\langle x+\tfrac{a}{N}\rangle) = B_k(\langle Nx\rangle), \qquad \text{for } x \in \mathbb{Q}/\mathbb{Z}.$$

PROPOSITION 3.1. B. MAZUR. *For each fixed k, $x \to N^{k-1}B_k(\langle\frac{x}{N}\rangle)$ defines on the projective system $\mathbb{Z}/N\mathbb{Z}$ a compatible family of distributions, hence also a distribution E_k on the limit $\hat{\mathbb{Z}}$* (see introduction, example (b)).

Let f be a function of period N on $\mathbb{Z}$, denote also by f the corresponding locally constant function on $\hat{\mathbb{Z}}$, and by f_N the function on $\mathbb{Z}/N\mathbb{Z}$, then $B_{k,f} = (E_k|f) = (E_k^{(N)}|f_N)$.

If f takes its values in a p-adic field (for instance, if it takes algebraic values which can be imbedded in a p-adic field, like a Dirichlet character does) then

$$B_{k,f} = \lim_{h\to+\infty} \left(\frac{1}{Np^h} \sum_{a=0}^{Np^h-1} f(a)a^k\right), \text{ where the limit is p-adic.}$$

PROPOSITION 3.2 Let E_k be the distribution on $\mathbb{Z}_p$ defined by the family $E_k^{(p^n)}$. Then for $k \geq 1$, $E_k = D^k(E_0)$ and

$$E_0(\theta) = (\mathrm{Log}\ \theta)/\theta-1.$$

For $k \geq 0$, E_k belongs to $(C_u^{(k)})'$.

Proof: According to lemma 2.4, if there exists an $F \in LA'$ and such that $c_n(F) = (E_n|1) = B_n$, for $n \geq 0$, then $G(X) = \Sigma c_n(F)X^n/n! = F(\exp X-1)$. Hence here necessarily, $G(X) = \Sigma B_n X^n/n! = X/e^X-1$, and $F(X) = \mathrm{Log}\ X/X-1$. This expression shows that $F \in (C_u^{(k)})'$, and the relation $E_k = D^k(E_0)$ is an easy consequence of 2.3.1 together with the classical expression of the polynomial $B_k(X)$ by means of the numbers B_n.

From here, one can give an interpretation of many classical results concerning L-functions, in terms of values of tempered distributions. As an example, one can get the following.

PROPOSITION 3.3 Let $c \in \mathbb{Q}$ be such that $(c,p) = 1$ and define, for $x \in \mathbb{Z}/N\mathbb{Z}$, $E_{k,c}^{(N)}(x)$ by $E_{k,c}^{(N)}(x) = E_k^{(N)}(x) - c^k E_k^{(N)}(c^{-1}x)$. Let M_1 be the (tempered) distribution on $\mathbb{Z}_p$ defined by the family of the $E_{1,c}^{(N)}$ for $N = p^h$, then $(M_1|t^k) = (1-c^{k+1})B_{k+1}/k+1$ and $M_1(X) = (1/X-1)-(c/X^c-1)$, so that M_1 is a measure.

In an analogous way, if χ is a Dirichlet character with conductor f, let $F_\chi(X) = \frac{1}{f}\sum_{a=0}^{f-1} \chi(a)\frac{X^a}{X^f-1} - \frac{c}{f}\sum_{a=0}^{f-1}\chi(ac)\frac{X^{ac}}{X^{fc}-1}$, then F_χ is the distribution "corresponding" to the L function of the character χ in the sense that $(E_\chi|t^n) = L_p(1-n,\chi)$, where L_p is the usual "corrected" L-function, described in [11],[10] for instance. Other functions, as described in [7] and [8] for instance, can be interpreted in terms of tempered distributions.

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