every rational function of $J,J_1,\ldots,J_\kappa$, and every integral algebraic function i of $J_1,\ldots,J_\kappa$ is an invariant of the ground form. For, since the function i depends rationally on the invariants $J,J_1,\ldots,J_\kappa$, it must be a rational function of the coefficients of the ground form: we put $i = g/h$, where g and h are polynomials in the coefficients of the ground form without a common factor. Further, i satisfies an equation of the form

$$i^k + G_1 i^{k-1} + \cdots + G_k = 0 \quad ,$$

where $G_1,\ldots,G_k$ are polynomials in $J_1,\ldots,J_\kappa$. Substituting $i = g/h$ into this equation and multiplying the resulting equation by h^{k-1} shows that g^k/h is a polynomial in the coefficients of the ground form. But since g and h are relatively prime, h must be a constant, i.e., i is a polynomial in the coefficients of the ground form and so is a polynomial invariant. From this follows the proposition:

The invariants $J,J_1,\ldots,J_\kappa$ determine a function field, in which the integral algebraic functions constitute all the polynomial invariants; in the sequel this function field will be called simply the field of invariants.

Now, by a fundamental proposition due to L. Kronecker, in any function field there is always a finite number of integral functions in terms of which any integral function in the field can be expressed as a linear combination whose coefficients are integral rational functions of the field; and the general theory of algebraic functions developed by L. Kronecker also shows how to determine the integral algebraic functions of a field. According to this, to recover the complete system of invariants $i_1,\ldots,i_m$ from the invariants $J,J_1,\ldots,J_\kappa$, one first computes the discriminant D of the equation of degree k for J. The invariants of the ground form, i.e., the integral algebraic functions in the field of invariants, are then all of the form

$$i = \frac{\Gamma_1 J^{k-1} + \Gamma_2 J^{k-2} + \cdots + \Gamma_k}{D} \quad .$$

Applying Theorem I of Section I of my work cited above* to the infinite sequence constructed from the functions $\Gamma_1, \Gamma_2, \ldots, \Gamma_k$, we see that there is a finite number $j_1,\ldots,j_M$ of invariants such that any invariant i can be represented as

$$i = A_1 j_1 + \cdots + A_M j_M \quad ,$$

where $A_1,\ldots,A_M$ are polynomials in $J_1,\ldots,J_\kappa$. The invariants $J_1,\ldots,J_\kappa$, $j_1,\ldots,j_M$ then form a complete system of invariants.

The construction of a complete system of invariants requires, in addition to the invariants $J,J_1,\ldots,J_\kappa$, only the solution of an elementary problem from the arithmetic theory of algebraic functions.

II. The Vanishing of the Invariants

3. A GENERAL THEOREM ON ALGEBRAIC FORMS

Since all the invariants of the ground form are integral algebraic functions of $J_1,\ldots,J_\kappa$, it follows immediately that

If the coefficients of the ground form are assigned special values so that the κ invariants $J_1,\ldots,J_\kappa$ all vanish, then so do all the invariants of the ground form.

It is now of the greatest significance for the theory to be developed here that this property of the system of invariants $J_1,\ldots,J_\kappa$ in fact characterizes these invariants. To prove this we first develop a theorem which represents a third general theorem from the theory of algebraic functions, continuing Theorems I and III of my work cited above.** This theorem states:

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*See this volume, page 143.

**See this volume, pages 143 and 168.

Let $f_1,\dots,f_m$ *be* m *homogeneous polynomials in the variables* $x_1,\dots,x_n$; *and let* $F,F',F'',\dots$ *be any homogeneous polynomials in the same variables which vanish for any values of the variables for which* $f_1,\dots,f_m$ *all vanish. Then one can always determine an integer* r *such that every product* $\Pi^{(r)}$ *of* r *arbitrary functions from the sequence* $F,F',F'',\dots$ *can be represented in the form*

$$\Pi^{(r)} = a_1 f_1 + a_2 f_2 + \cdots + a_m f_m \quad ,$$

where $a_1,a_2,\dots,a_m$ *are appropriately chosen homogeneous polynomials in* $x_1,\dots,x_n$.

In the following proof of this theorem we first assume that the sequence of forms $F,F',F'',\dots$ consists only of a *finite* number of forms.

The proof falls into two parts: in the first part we prove the theorem in the special case where the m given forms $f_1,\dots,f_m$ have only a finite number of common zeros. To carry out this proof we assume that the theorem has already been seen to hold for a certain number of common zeros and we then show that it also holds for forms with an additional common zero.

Let the common zeros of the forms $f_1,\dots,f_m$ be

$$x_1 = \alpha_1,\ x_2 = \alpha_2,\ \dots,\ x_n = \alpha_n,$$

$$x_1 = \beta_1,\ x_2 = \beta_2,\ \dots,\ x_n = \beta_n,$$

$$\cdots\cdots\cdots\cdots$$

$$x_1 = \kappa_1,\ x_2 = \kappa_2,\ \dots,\ x_n = \kappa_n \quad .$$

We now replace the variables $x_1,\dots,x_n$ by the expressions $x_1\xi_1, x_2\xi_1, \dots, x_{n-1}\xi_1, \xi_2$, whereby the forms $f_1,\dots,f_m$ are transformed into binary forms in the variables ξ_1,ξ_2 of degrees $\nu_1,\dots,\nu_m$. Then we construct the expressions

$$F_1 = u_1 f_1 + u_2 f_2 + \cdots + u_m f_m \quad ,$$

$$F_2 = v_1 f_1 + v_2 f_2 + \cdots + v_m f_m \quad ,$$

where $u_1,\dots,u_m$, $v_1,\dots,v_m$ are binary forms with undetermined coefficients and of such degrees in the variables ξ_1,ξ_2 that F_1 and F_2 may be homogeneous in ξ_1,ξ_2. The resultant of the two binary forms F_1,F_2 with respect to the variables ξ_1,ξ_2 then becomes a polynomial in the undetermined coefficients of the forms $u_1,\dots,u_m$, $v_1,\dots,v_m$, and the powers and products of these undetermined coefficients occur multiplied by forms which contain only the $n-1$ variables $x_1,\dots,x_{n-1}$; denote these forms by $f_1',\dots,f_{m'}'$. From the properties of the resultant of two binary forms one sees easily that the forms $f_1',\dots,f_{m'}'$ have only the following zeros in common:

$$x_1 = \alpha_1,\ x_2 = \alpha_2,\ \dots,\ x_{n-1} = \alpha_{n-1} \quad ,$$

$$\cdots\cdots\cdots\cdots$$

$$x_1 = \kappa_1,\ x_2 = \kappa_2,\ \dots,\ x_{n-1} = \kappa_{n-1} \quad ,$$

and that, moreover, these forms are all linear combinations of the forms $f_1,\dots,f_m$, i.e.,

$$\left.\begin{aligned} f_1' &\equiv 0 \\ &\cdots \\ f_{m'}' &\equiv 0 \end{aligned}\right\} \ (f_1,\dots,f_m) \quad .$$

Reapplying this elimination procedure, this time to the forms $f_1',\dots,f_{m'}'$, we obtain a system of forms $f_1'',\dots,f_{m''}''$ in the $n-2$ variables $x_1,\dots,x_{n-2}$ which have, as common zeros, only

$$x_1 = \alpha_1,\ x_2 = \alpha_2,\ \dots,\ x_{n-2} = \alpha_{n-2}$$

$$\cdots\cdots\cdots\cdots$$

$$x_1 = \kappa_1,\ x_2 = \kappa_2,\ \dots,\ x_{n-2} = \kappa_{n-2} \quad ,$$

and which are all congruent to zero with respect to the module $(f_1',\ldots,f_m')$ and hence also with respect to the module $(f_1,\ldots,f_m)$. Continuing this process yields finally a system of binary forms $f_1^{(n-2)},\ldots,f_m^{(n-2)}$ in the variables x_1,x_2 which have, as common zeros, only

$$x_1 = \alpha_1,\quad x_2 = \alpha_2,$$

$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$

$$x_1 = \kappa_1,\quad x_2 = \kappa_2$$

and which are all congruent to zero with respect to the module $(f_1,\ldots,f_m)$. We choose one of these binary forms and set it equal to $(\alpha_1x_1-\alpha_1x_2)^{\rho_{12}}\phi_{12}$, where ρ_{12} is a positive integer and ϕ_{12} is a binary form which vanishes for $x_1 = \alpha_1$, $x_2 = \alpha_2$. Here it is assumed that the quantities α_1,α_2 are not both zero.

In the same way we find, when α_1,α_3 are not both zero, that there is an integer ρ_{13} and a binary form ϕ_{13} in the variables x_1,x_3 which does not vanish for $x_1 = \alpha_1$, $x_3 = \alpha_3$ and such that

$$(\alpha_3x_1-\alpha_1x_3)^{\rho_{13}}\phi_{13} \equiv 0 \pmod{(f_1,\ldots,f_m)}$$

And finally, let $\rho_{n-1,n}$ be an integer and $\phi_{n-1,n}$ a binary form in the variables x_{n-1},x_n which does not vanish for $x_{n-1} = \alpha_{n-1}$, $x_n = \alpha_n$ and such that

$$(\alpha_nx_{n-1}-\alpha_{n-1}x_n)^{\rho_{n-1,n}}\phi_{n-1,n} \equiv 0 \pmod{(f_1,\ldots,f_m)}$$

Since, by hypothesis, every form in the sequence $F,F',F'',\ldots$ vanishes for $x_1 = \alpha_1$, $x_2 = \alpha_2,\ldots,$ $x_n = \alpha_n$, one can put

$$F^{(i)} = F_{12}^{(i)}(\alpha_2x_1-\alpha_1x_2) + F_{13}^{(i)}(\alpha_3x_1-\alpha_1x_3) + \cdots + F_{n-1,n}^{(i)}(\alpha_nx_{n-1}-\alpha_{n-1}x_n)$$

where $F_{12}^{(i)}, F_{13}^{(i)},\ldots,F_{n-1,n}^{(i)}$ are forms in the n variables $x_1,\ldots,x_n$.

From this it follows, using the above congruences, that if one puts, for brevity,

$$\rho = \rho_{12} + \rho_{13} + \cdots + \rho_{n-1,n},$$

and

$$\Phi = \phi_{12}\phi_{13}\cdots\phi_{n-1,n},$$

then

$$\Phi\Pi^{(\rho)} \equiv 0 \pmod{(f_1,\ldots,f_m)},$$

where Φ is a form which does not vanish for $x_1 = \alpha_1$, $x_2 = \alpha_2,\ldots,$ $x_n = \alpha_n$ and $\Pi^{(\rho)}$ is the product of any ρ of the forms in the sequence $F,F',F'',\ldots$

The forms $\Phi, f_1,\ldots,f_m$ have fewer common zeros than the forms $f_1,\ldots,f_m$ of the original system. Hence if we assume the theorem correct for systems of forms with fewer common zeros, it follows that there is a number r such that

$$\Pi^{(r)} \equiv \bullet \pmod{(\Phi, f_1,\ldots,f_m)},$$

where $\Pi^{(r)}$ is the product of any r of the forms in the sequence $F,F',F'',\ldots$ From this it follows, using the above congruences, that

$$\Pi^{(\rho+r)} \equiv 0 \pmod{(f_1,\ldots,f_m)},$$

where $\Pi^{(\rho+r)}$ is the product of any $\rho+r$ functions in the sequence $F,F',F'',\ldots$. This proves that the theorem holds for the system of forms $f_1,\ldots,f_m$ under the stated hypothesis.

Now, the theorem holds in the case where the given forms have *no* common zeros. For in this case the binary forms $f_1^{(n-2)},\ldots,f_m^{(n-2)}$ have no common zeros; hence every binary form in x_1,x_2 of sufficiently high degree, in particular the forms $x_1^{\rho_1}$ and $x_2^{\rho_2}$ for sufficiently large

exponents ρ_1 and ρ_2, is congruent to 0 with respect to the module $(f_1,\dots,f_m)$. In the same way one shows that for sufficiently large exponents $\rho_3,\dots,\rho_n$ the forms $x_3^{\rho_3},\dots,x_n^{\rho_n}$ are congruent to 0 with respect to the module $(f_1,\dots,f_m)$. It follows that any form in the variables $x_1,\dots,x_n$ of degree $\geq \rho_1+\rho_2+\cdots+\rho_n$ is $\equiv 0$ with respect to the module $(f_1,\dots,f_m)$; this proves the above assertion.

In the *second* part the theorem is proved, in general. For this purpose we assume that it has been seen to be true for arbitrary forms in $n-1$ variables, and we show that it is also true for n variables.

If we put $x_1 = tx_2$, the forms $f_1,\dots,f_m,F,F',\dots$ go over into forms in the $n-1$ variables $x_2,\dots,x_n$ whose coefficients are polynomials in the parameter t. Denote these forms by $g_1,\dots,g_m,G,G',\dots$, respectively. If we now assign to the parameter t an arbitrary definite finite value, it is evident that each form in the sequence $G,G',\dots$ vanishes for those values of the variables $x_2,\dots,x_n$ which make all the m forms $g_1,\dots,g_m$ vanish. Now assume the theorem proved for the case of $n-1$ variables, and let it also be assumed that in this case one can choose the number r below a certain bound which depends only on the degrees and the number of the forms $g_1,\dots,g_m,G,G',\dots$ and not on their coefficients. Then we know that there is a number $r = \sigma_{12}$ such that any product $\Pi^{(\sigma_{12})}$ of σ_{12} of the forms in the sequence $G,G',\dots$ for each special value of t admits a representation as

$$\Pi^{(\sigma_{12})} = b_1g_1 + b_2g_2 + \cdots + b_mg_m \quad ,$$

where $b_1,\dots,b_m$ are homogeneous polynomials in the $n-1$ variables $x_2,\dots,x_n$. If we consider the coefficients u of the forms $b_1,\dots,b_m$ in this formula as undetermined quantities and then compare the coefficients of the same powers and products of the variables $x_2,\dots,x_n$ on the two sides, we obtain a non-homogeneous system of linear equations for determining the coefficients u. In these linear equations the coefficients are polynomials in the parameter t, and we know besides that this system of linear equations has solutions for every particular finite value of t.

We will use the following lemma, which is easy to prove:

If a system of linear equations of the form

$$c_{11}u_1 + \cdots + c_{1p}u_p = c_1 \quad ,$$

$$\cdots\cdots\cdots$$

$$c_{q1}u_1 + \cdots + c_{qp}u_p = c_q$$

is given, where $c_{11},c_{12},\dots,c_{qp},c_1,\dots,c_q$ are polynomials in a parameter t, and has solutions for every particular value of t, then one can always determine rational functions of t which, when substituted for the unknowns $u_1,\dots,u_p$, transform the above equations into identities in the parameter t.

Applying this lemma to the equations obtained above and then putting $t = x_1/x_2$, and then clearing denominators, we obtain a congruence of the form

$$\psi_{12}\Pi^{(\sigma_{12})} \equiv 0 \quad , \qquad (f_1,\dots,f_m) \quad ,$$

where ψ_{12} is a binary form in the two variables x_1,x_2, and where $\Pi^{(\sigma_{12})}$ is a product of any σ_{12} of the forms in the sequence $F,F',\dots$

In the same way we obtain a congruence of the form

$$\psi_{13}\Pi^{(\sigma_{13})} \equiv 0 \quad , \qquad (f_1,\dots,f_m) \quad ,$$

where ψ_{13} is a binary form in the two variables x_1,x_3, and where $\Pi^{(\sigma_{13})}$ is a product of σ_{13} of the forms $F,F',\dots$ Finally, let $\sigma_{n-1,n}$ be an integer and $\psi_{n-1,n}$ a form in the two variables x_{n-1},x_n such that the congruence

$$\psi_{n-1,n}\Pi^{(\sigma_{n-1,n})} \equiv 0 \quad , \qquad (f_1,\dots,f_m)$$

holds. Now since it is evident that there is only a finite number of systems of values for which the forms $\psi_{12},\psi_{13},\dots,\psi_{n-1,n},f_1,\dots,f_m$ all vanish, this system of forms is of the kind for which the theorem has been

assumed to hold. Hence a number r can be found so that

$$\Pi^{(r)} \equiv 0 \quad , \qquad (\psi_{12},\psi_{13},\dots,\psi_{n-1,n},f_1,\dots,f_m) \quad .$$

From this it follows, using the above congruences, that

$$\Pi^{(\sigma+r)} \equiv 0 \quad , \qquad (f_1,\dots,f_m) \quad ,$$

where σ is the largest of the numbers $\sigma_{12},\sigma_{13},\dots,\sigma_{n-1,n}$.

Since binary forms can have only finitely many zeros anyway, by the first part of the proof the theorem is true in the special case $n = 2$ and therefore also in general for forms in n variables. Now if the given series $F,F',\dots$ contains infinitely many forms, then one determines a number μ such that every form in the sequence $F,F',\dots$ is a combination of the μ forms $F,F',\dots,F^{(\mu-1)}$; this is always possible by Theorem I of my work cited above.* Now if the product of any r of the forms $F,F',\dots,F^{(\mu-1)}$ is $\equiv 0$ with respect to the module $(f_1,\dots,f_m)$, then obviously the same is true for any product of r forms in the sequence $F,F',\dots$; and this completes the proof of the theorem.

By the theorem just proved the r-th power of any of the forms $F,F',F'',\dots$ is 0 with respect to the module $(f_1,f_2,\dots,f_m)$; in the special case of two non-homogeneous variables this has been announced and proved by E. Netto.**

4. THE FUNDAMENTAL THEOREM ON THE INVARIANTS WHOSE VANISHING IMPLIES THE VANISHING OF ALL INVARIANTS

We now resume the developments in the theory of the invariants of a ground form or of a system of ground forms which were interrupted at the beginning of the preceding section, and we prove the following fundamental proposition:

If any μ invariants $I_1,\dots,I_\mu$ have the property that their vanishing implies the vanishing of all invariants, then all the invariants are integral algebraic functions of $I_1,\dots,I_\mu$.

By hypothesis the μ invariants $I_1,\dots,I_\mu$ are functions of the coefficients of the ground form such that whenever these coefficients are assigned particular values which make $I_1,\dots,I_\mu$ vanish, all the invariants of the ground form vanish. Hence it follows from the general theorem of Section 3 that there is a number r such that every product $\Pi^{(r)}$ of any r or more invariants of the ground form can be represented as

$$\Pi^{(r)} = a_1I_1 + a_2I_2 + \cdots + a_\mu I_\mu \quad ,$$

where $a_1,a_2,\dots,a_\mu$ are homogeneous polynomials in the coefficients of the ground form. As before, we denote by $i_1,\dots,i_m$ a complete system of invariants; and let ν be the maximum degree of these invariants. Then it is evident that any invariant i of the ground form of degree $\geq \nu r$ can be represented as a sum of products $\Pi^{(r)}$, and so

$$i = a_1'I_1 + a_2'I_2 + \cdots + a_\mu'I_\mu \quad ,$$

where $a_1',a_2',\dots,a_\mu'$ are again homogeneous polynomials in the coefficients of the ground form. By the developments of my paper, "On the Theory of Algebraic Forms"* cited above, one can replace the expressions $a_1',a_2',\dots,a_\mu'$ by invariants $i_1',i_2',\dots,i_\mu'$ so that

$$i = i_1'I_1 + i_2'I_2 + \cdots + i_\mu'I_\mu \quad .$$

The invariants $i_1',i_2',\dots,i_\mu'$ are all of lower degree in the coefficients of the ground form than i. In the same way they can be replaced by linear combinations of the invariants $I_1,I_2,\dots,I_\mu$, and this process can be continued until we obtain invariants of degree $< \nu r$. Let $j_1,j_2,\dots,j_w$ be all linearly independent invariants of degree $< \nu r$. Then for an

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* [See this volume, page 143.]

** Cf. Acta math. vol. 7, p. 101.

* [See this volume, page 216.]