KRASNER’S ANALYTIC FUNCTIONS AND RIGID ANALYTIC SPACES

by

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At present, we have two general theories of analytic functions over a non-trivial non-archimedean field. One is Krasner’s theory of analytic functions and the other is Tate’s theory of rigid analytic spaces. The main purpose of this paper is to study the relation between them.

§1 is a preparation from Krasner’s theory. The main results will be given in §2.

We shall show that (1) if \( k \) is algebraically closed and maximally complete and \( D \) is a completely regular quasi-connected set in \( \mathbb{P}^1(k) \), then the both theories give the same results, but (2) if \( k \) or \( D \) does not satisfy these conditions, then they give different results.

§1. Krasner’s analytic functions

Let \( k \) be an algebraically closed field, \( | \cdot | \) a non-trivial non-archimedean valuation on \( k \). We assume that \( k \) is complete with respect to \( | \cdot | \). Let \( \mathbb{P}^1(k) = k \cup \{\infty\} \) be the one dimensional projective space over \( k \). Then \( \mathbb{P}^1(k) \) has a natural structure of a metric space. We extend \( | \cdot | \) to \( \mathbb{P}^1(k) \) by \( |\infty| = +\infty \).

Let \( D \) be an open subset of \( \mathbb{P}^1(k) \). We say that \( D \) is quasi-connected if for any two points \( x, y \) of \( D \) such that \( x \neq y \), the set

\[ \{ |x-z|; z \in \mathbb{P}^1(k), z \not\in D, |x-z| \leq |x-y| \} \]

is a finite set. Let \( \mathcal{D} \) be a family of quasi-connected sets. Then we say that \( \mathcal{D} \) is a chain if for any two elements \( D \) and \( D' \) of \( \mathcal{D} \), there exists a finite number of elements \( D'_1, \ldots, D'_n \) of \( \mathcal{D} \) such that \( D_i = D, D_i \cap D_{i+1} \neq \emptyset (i = 1, \ldots, n-1), D_n = D' \). It is known that (1) any linear fractional transformation of \( \mathbb{P}^1(k) \) maps a quasi-connected set to a quasi-connected set.

(2) the intersection of any finite number of quasi-connected sets is either empty or quasi-connected, and (3) the union of a chain of quasi-connected sets is quasi-connected.

Let \( D \) be a quasi-connected set. Then we denote by \( \mathcal{O}_D(D) \) the set consisting of all functions \( f : D \rightarrow k \) such that there exists a sequence \( \{ f_n \}_{n=1}^{\infty} \) of rational functions with coefficients in \( k \) such that the \( f_n \) have no pole in \( D \) and \( f_n \rightarrow f \) uniformly on \( D \).

Let \( \mathcal{D} = \{ D_1 \}_{\alpha \in \mathcal{I}} \) be a chain of quasi-connected sets. Then \( \mathcal{D} \) gives an open covering of \( D = \bigcup_{\alpha \in \mathcal{I}} D_1 \). We see that the family of all such coverings determines a Grothendieck topology \( J_\mathcal{I} \) and the functor \( D \rightarrow \mathcal{O}_D(D) \) is a presheaf with respect to this Grothendieck topology. Let \( \mathcal{O} \) be the sheafication of \( \mathcal{O}_0 \). For any quasi-connected set \( D \), an element of \( \mathcal{O}(D) \) (resp. an element of \( \mathcal{O}_0(D) \)) is called an analytic function on \( D \) (resp. an analytic element of support \( D \)).

In a series of his paper M. Krasner defined and studied the properties of analytic functions on quasi-connected sets (cf e.g. Krasner [4]). For example, he proved the theorem of identity for such functions. Later P. Robba generalized the theory to include a wider class of open subsets of \( \mathbb{P}^1(k) \) (cf. Robba [7]). Owing to them, we know the structure of an analytic element very well. On the other hand, the author restricted the attention to a narrower class of open subsets of \( \mathbb{P}^1(k) \) and studies the properties of analytic functions more closely (cf. Morita [5]).
EXAMPLE 1. Let \( D = \{ z \in \mathbb{P}^1(k) \mid |z-a_i| \leq r_i \} \) (\( i = 1, \ldots, n \)), where \( a_i \in k \) and \( r_i \in |k^r| \). Then \( D \) is a quasi-connected set and any element \( f \) of \( \mathcal{O}_D \) can be expressed in the form

\[
f(z) = c_\infty + \sum_{i=1}^{n} \sum_{m=1}^{\infty} c_m (z-a_i)^m.
\]

It is known that if \( f \) is maximally complete (i.e. any decreasing sequence \( c_1 \supseteq c_2 \supseteq \ldots \supseteq c_n \supseteq \ldots \) of balls in \( k \) satisfies \( \bigcap_n c_n \neq \emptyset \)), then \( \mathcal{O}_D = \mathcal{O}_D \) (cf. Krasner [4]). But if \( f \) is not maximally complete, then we can prove that \( \mathcal{O}_D \not\supseteq \mathcal{O}_D \). Hereafter we say that an open subset \( D' \) of \( \mathbb{P}^1(k) \) is a connected (open) affine subset of \( \mathbb{P}^1(k) \) if there exists a linear fractional transformation \( g \) of \( \mathbb{P}^1(k) \) such that \( g(D') \) has the above form.

Let \( D \) be a quasi-connected set, \( C \) a closed ball of the form \( \{ z \in k \mid |z-a| \leq r \} \) (\( a \in k, r \in |k^r| \)) satisfying \( r \leq \sup_{x,y \in C} |x-y| \) and \( C \cap D \neq \emptyset \). Then \( C \) can be decomposed into a disjoint union of open balls \( C_B \) of the form \( \{ z \in k \mid |z-b| < r \} \). We say that \( D \) is completely regular if for any such closed ball \( C \), all but a finite number of such open balls \( C_B \) are contained in \( D \). It is easy to see that (1) this property is preserved under the action of the linear fractional group, (2) it is also preserved by a finite non-empty intersection and by a union of a chain, and (3) a connected affine subset of \( \mathbb{P}^1(k) \) is completely regular. Furthermore we obtained in [5]

THEOREM 1. Let \( k \) be maximally complete, \( D \) a completely regular quasi-connected set \( \not\supseteq \mathbb{P}^1(k) \). Then there exists a sequence

\[
(D_n)^m \quad n=1
\]

of connected open affine subsets of \( \mathbb{P}^1(k) \) such that

\[
D_1 \subseteq D_2 \subseteq \ldots \subseteq D_n \subseteq \ldots \quad \text{and} \quad D = \bigcup_{n=1}^{\infty} D_n.
\]

Furthermore any two such sequences are cofinal.

§2. Relation between completely regular quasi-connected sets and rigid analytic spaces

As in §1, Krasner used Runge's theorem to construct his theory of analytic functions. On the other hand, Tate constructed his theory by adding one structure on analytic spaces:

Let \( k \) be as in §1. Let \( k[t_1, \ldots, t_n] \) be the ring consisting of all power series \( \sum_{m_1, \ldots, m_n \geq 0} c_{m_1, \ldots, m_n} t_1^{m_1} \ldots t_n^{m_n} \) such that \( m_1, \ldots, m_n \to 0 \) for \( m_1 + \ldots + m_n \to \infty \). We say that a \( k \)-algebra \( A \) is \text{tft} over \( k \) if \( A \) is isomorphic to a quotient of \( k[t_1, \ldots, t_n] \).

Let \( A \) be the category of all such \( k \)-algebras. For any \( A \in A \), let \( \text{Max}(A) \) be the set of maximal ideals of \( A \). Then \( \text{Max}(A) \) has a natural structure of an analytic space over \( k \), and any analytic space over \( k \) is locally isomorphic to some \( \text{Max}(A) \) \( (A \in A) \).

Let \( X \) be an analytic space over \( k \). Then an \( h \)-structure on \( X \) is a selection, for each \( A \in A \), of a certain subset of \( k \)-ranged space morphisms \( \text{Max}(A) \to X \) satisfying two conditions (h1) and (h2). An \( h \)-space over \( k \) is an analytic space over \( k \) with an \( h \)-structure, and a rigid analytic space over \( k \) is an \( h \)-space over \( k \) which has a "good" affine covering. (For more details, see Tate [8].)

Now we are going to study the relation between these two theories. We quote the following result from [6].

THEOREM 2. Let \( k \) be as in §1, \( A \) tft. We assume that \( A \) is an integral domain. Then for any element \( f \) of \( A \), the image

\[
\text{Max}(A) \ni p \mapsto f(p) \in k
\]

is a point or a connected open affine subset of \( \mathbb{P}^1(k) \).
Let $D$ be a connected open affine subset of $\mathbb{P}^1(k)$. Then $\mathcal{O}_D(D)$ is ttf over $k$ and the space $\text{Max}(\mathcal{O}_D(D))$ is canonically isomorphic to $D$. Hence $D$ has a natural structure of a rigid analytic space, and $\mathcal{O}_D(D)$ can be identified with the set of all $h$-morphisms of $D$ to $k$ (cf. Tate [8]).

Let $A$ be ttf over $k$. We say that a map $f$ of $\text{Max}(A)$ to $\mathbb{P}^1(k)$ is structural if there exists a connected open affine subset $D$ of $\mathbb{P}^1(k)$ such that $f$ is a composite of an $h$-morphism of $\text{Max}(A)$ to $D$ and the inclusion map $D \hookrightarrow \mathbb{P}^1(k)$. Then this defines a structure of a rigid analytic space on $\mathbb{P}^1(k)$ (cf. Tate [8], Proposition 10.15).

Now we assume that $k$ is maximally complete. Let $D$ be a completely regular quasi-connected set $\subseteq \mathbb{P}^1(k)$, $\{D_n\}_{n=1}^\infty$ as in Theorem 1. Since $D$ is an open subset of the $h$-space $\mathbb{P}^1(k)$, $D$ has an $h$-structure (cf. Tate [8], p.282). Let $A$ be ttf over $k$, and let $f$ be an $h$-morphism of $\text{Max}(A)$ to $D$. Then it follows Theorem 2 and the definition of the $h$-structure of $D$ that the image of $f$ is a finite union of points and connected open affine subsets contained in $D$.

It follows from Proposition 2.5 of [5] that the image of $f$ is contained in $D_n$ for any sufficiently large $n$. This shows that $\{D_n\}_{n=1}^\infty$ is an admissible covering of $D$ and the $h$-structure of $D$ can be obtained by gluing up the natural $h$-structures of the $D_n$'s (cf. Tate [8], Proposition 10.15). Hence $D$ is a quasi-Stein space of Kiehl [3]. Furthermore a map $f$ of $D$ to $k$ is an $h$-morphism iff the restriction of $f$ to $D_n$ is always an $h$-morphism (cf. Tate [8], Proposition 10.14). Hence $f$ is an $h$-morphism iff the restriction of $f$ to $D_n$ is always an analytic function of Krasner (cf. Example 1). Therefore a map of $D$ to $k$ is an $h$-morphism iff $f$ is an analytic function in the sense of Krasner.

Let $\mathcal{D} = \{D^{(m)}_n\}_{n=1}^\infty$ be a chain of a countable number of completely regular quasi-connected sets satisfying $\bigcup_{n=1}^\infty D^{(m)}_n = D$.

By renumbering, we may assume $D^{(m)}_n \cap \bigcup_{m=1}^{N-1} D^{(m)}_m \neq \emptyset$ for any $M \geq 1$.

Let $\{D^{(m)}_n\}_{n=1}^\infty$ be as in Theorem 1. Here we may assume that $\bigcap_{M=1}^\infty \bigcup_{n=1}^\infty D^{(m)}_n$ is not empty. Since $\bigcup_{n=1}^\infty \{D^{(m)}_n\}_{n=1}^\infty$ is a refinement of $\{D^{(m)}_n\}_{n=1}^\infty$, this shows that $\mathcal{D}$ is an admissible covering of $D$. Hence, by the result of Kiehl [4], $\text{H}^p(D, \mathcal{O}) = 0$ for any positive integer $p$.

Therefore we have proved

**THEOREM 3.** Let $k$ be maximally complete, $D$ a completely regular quasi-connected set $\subseteq \mathbb{P}^1(k)$, $\{D_n\}_{n=1}^\infty$ as in Theorem 1. Then

1. The natural $h$-structure of $\mathbb{P}^1(k)$ induces on $D$ a structure of a quasi-Stein space. More closely, this structure can be obtained by gluing up the natural $h$-structure of the $D_n$'s.

2. $\mathcal{O}(D)$ can be identified with the set of all $h$-morphisms of $D$ to $k$.

3. Let $\mathcal{D} = \{D^{(m)}_n\}_{n=1}^\infty$ be a chain of a countable number of completely regular quasi-connected sets satisfying $\bigcup_{n=1}^\infty D^{(m)}_n = D$. Then $\mathcal{D}$ is an admissible covering of $D$. In particular $\text{H}^p(D, \mathcal{O}) = 0$ for any positive integer $p$.

Now we assume that $k$ is not maximally complete. Then there exists a decreasing sequence $C_1 \supseteq C_2 \supseteq \ldots \supseteq C_n \supseteq \ldots$ of open balls in $k$ such that the diameter of each $C_n$ belongs to $[k^2]$ and $\bigcap_{n=1}^\infty C_n \neq \emptyset$.

Then $D_n = \mathbb{P}^1(k) \setminus D_n$ is a connected open affine subset of $\mathbb{P}^1(k)$. Hence,
by gluing up the natural $h$-structures of the $D_{a,b}$'s, we can construct
on $p^1(k)$ a structure of a quasi-Stein space. Of course, this $h$
structure is different from the natural $h$-structure of $p^1(k)$.

It is easy to see that any $h$-morphism of this quasi-Stein space
$p^1(k)$ to $k$ is an analytic function in the sense of Krasner.

Let $k$ be as above. Let $D$ be a completely regular quasi-connected
set, $D_{a,b}$ $(a,b \in D)$ as in [5], 1-4. Then $D_{a,b}$ is a connected open
affine subset of $p^1(k)$ and $\bar{D} = \overline{D_{a,b}}$ is a chain. Furthermore,
it follows from Theorem 2 that $\bar{D}$ is an admissible covering of $D$.

Hence a mapping of $D$ to $k$ is an $h$-morphism iff the restriction of it
to each $D_{a,b}$ is an $h$-morphism. Therefore, in view of the definition
of the $h$-structure of $D_{a,b}$, such a mapping is an analytic function
in the sense of Krasner.

Let $k$ be as in 11 and let $D$ be any open subset of $p^1(k)$. For
any $x, y \in D$, we write $x \sim y$ if there exists a completely regular
$\omega_{x,b}$ of $D$ containing $x$ and $y$. Then this $\sim$ is an equi-
valence relation. Let $D = \bigsqcup_{1} D_{1}$ be the corresponding decomposition
of $D$.

Then it follows from Theorem 2 that (1) each $D_{1}$ is a maximal com-
pletely regular quasi-connected subset of $D$, (2) $\bar{D} = \overline{D_{1}}$ is an
admissible covering of the $h$-space $D$, and (3) a mapping of $D$ to
$k$ is an $h$-morphism iff the restriction of $f$ to each $D_{1}$ is an
$h$-morphism.

EXAMPLE 2. Let $k$ be maximally complete, $D = \{z \in k \mid 1/2 < |z| < 2,
|z| \neq 1\}$. Then $D$ is a quasi-connected set. Hence $\mathcal{O}(D)$ is an integral
domain.

On the other hand, the set of all $h$-morphisms of $D$ to $k$ is the
direct sum of $\mathcal{O}(\{z \in k \mid 1/2 < |z| < 1\})$ and $\mathcal{O}(\{z \in k \mid 1 < |z| < 2\})$.

FINITE-DIMENSIONAL NORMED $K$-VECTORSAPCES

by

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§0. Introduction

The field $K$ is supposed to be complete with respect to a non-Archimedean valuation. A norm $|| \cdot ||$ on a finite-dimensional vector-space $V$ over $k$ is called split if $V$ has a base $e_1, \ldots, e_n$ such that $|| \sum \lambda_i e_i || = \max \{ |\lambda_i| ||e_i|| \}$ for all $\lambda_1, \ldots, \lambda_n \in k$.

The normed vector-space $(V, || \cdot ||)$ is also called split and $(e_1, \ldots, e_n)$ is called an orthogonal base of $V$. It is well known (for Banach spaces over $k$ the reader can consult [2], [4] and [5]) that for a maximally complete field $k$ every norm on a finite-dimensional vector-space is split. So our main interest lies in the case where the base field $k$ is not maximally complete. The work in this paper is centered around the following questions:

Q0. What are the norms on a finite-dimensional vector-space?
Q1. What group of isometries does a finite-dimensional normed vector-space have?

This setup is of course related to $K$-theory where $K_0(R)$ describes the projective modules over $R$ and $K_1(R)$ describes the automorphism groups of projective modules. We will use very little $K$-theory (the reader may consult [1] or [3]). In §1 some $K_0$'s are discussed and finally in §1.4 finite-dimensional normed spaces over $k$ are classified by (non-trivial) linear algebra over $\overline{k}$ = the residue field of $k$. In §2 the group of isometries of a normed finite-dimensional vector-space