KRASNER'S ANALYTIC FUNCTIONS AND RIGID ANALYTIC SPACES

by

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At present, we have two general theories of analytic functions over a non-trivial non-archimedean field. One is Krasner's theory of analytic functions and the other is Tate's theory of rigid analytic spaces. The main purpose of this paper is to study the relation between them.

§1 is a preparation from Krasner's theory. The main results will be given in §2.

We shall show that (1) if k is algebraically closed and maximally complete and D is a completely regular quasi-connected set in $P^1(k)$, then the both theories give the same results, but (2) if k or D does not satisfy these conditions, then they give different results.

§1. Krasner's analytic functions

Let k be an algebraically closed field, $|\ |$ a non-trivial non-archimedean valuation on k. We assume that k is complete with respect to $|\ |$. Let $P^1(k) = k \cup \{\infty\}$ be the one dimensional projective space over k. Then $P^1(k)$ has a natural structure of a metric space. We extend $|\ |$ to $P^1(k)$ by $|\infty| = +\infty$.

Let D be an open subset of $P^1(k)$. We say that D is quasi-connected if for any two points x, y of D such that $x \neq \infty$, the set

$$\{|x-z| \mid z \in P^1(k),\ z \notin D,\ |x-z| \leqq |x-y|\}$$

is a finite set. Let $\mathcal{D}$ be a family of quasi-connected sets. Then

we say that $\mathcal{D}$ is a chain if for any two elements D and D' of $\mathcal{D}$, there exists a finite number of elements $D_1,\ldots,D_n$ of $\mathcal{D}$ such that $D_1 = D$, $D_i \cap D_{i+1} \neq \emptyset$ $(i = 1,\ldots,n-1)$, $D_n = D'$. It is known that

(1) any linear fractional transformation of $P^1(k)$ maps a quasi-connected set to a quasi-connected set,

(2) the intersection of any finite number of quasi-connected sets is either empty or quasi-connected, and

(3) the union of a chain of quasi-connected sets is quasi-connected.

Let D be a quasi-connected set. Then we denote by $\mathcal{O}_0(D)$ the set consisting of all functions $f : D \to k$ such that there exists a sequence $\{f_n\}_{n=1}^{\infty}$ of rational functions with coefficients in k such that the f_n have no pole in D and $f_n \to f$ uniformly on D.

Let $\mathcal{D} = \{D_i\}_{i\in I}$ be a chain of quasi-connected sets. Then $\mathcal{D}$ gives an open covering of $D = \bigcup_{i\in I} D_i$. We see that the family of all such coverings determines a Grothendieck topology $\mathcal{J}_q$ and the functor $D \to \mathcal{O}_0(D)$ is a presheaf with respect to this Grothendieck topology. Let $\mathcal{O}$ be the sheafsication of $\mathcal{O}_0$. For any quasi-connected set D, an element of $\mathcal{O}(D)$ (resp. an element of $\mathcal{O}_0(D)$) is called an analytic function on D (resp. an analytic element of support D).

In a series of his paper M. Krasner defined and studied the properties of analytic functions on quasi-connected sets (cf e.g. Krasner [4]). For example, he proved the theorem of identity for such functions. Later P. Robba generalized the theory to include a wider class of open subsets of $P^1(k)$ (cf. Robba [7]). Owing to them, we know the structure of an analytic element very well. On the other hand, the author restricted the attention to a narrower class of open subsets of $P^1(k)$ and studies the properties of analytic functions more closely (cf. Morita [5]).

EXAMPLE 1. Let $D = \{z \in P^1(k) \mid |z-a_i| \geqq r_i \ (i = 1,\ldots,n)\}$, where $a_i \in k$ and $r_i \in |k^\times|$. Then D is a quasi-connected set and any element f of $\mathcal{O}_0(D)$ can be expressed in the form

$$f(z) = c_\infty + \sum_{i=1}^{n} \sum_{m=-1}^{-\infty} c_m^{(i)} (z-a_i)^m.$$

It is known that if k is maximally complete (i.e. any decreasing sequence $C_1 \supseteqq C_2 \supseteqq \cdots \supseteqq C_n \supseteqq \cdots$ of balls in k satisfies $\bigcap_{m=1}^{\infty} C_m \neq \emptyset$), then $\mathcal{O}_0(D) = \mathcal{O}(D)$ (cf. Krasner [4]). But if k is not maximally complete, then we can prove that $\mathcal{O}_0(D) \subsetneqq \mathcal{O}(D)$. Hereafter we say that an open subset D' of $P^1(k)$ is a connected (open) affine subset of $P^1(k)$ if there exists a linear fractional transformation g of $P^1(k)$ such that g(D') has the above form.

Let D be a quasi-connected set, C a closed ball of the form $\{z \in k \mid |z-a| \leqq r\}$ ($a \in k$, $r \in |k^\times|$) satisfying $r \leqq \sup_{x,y \in D} |x-y|$ and $C \cap D \neq \emptyset$. Then C can be decomposed into a disjoint union of open balls C_b of the form $\{z \in k \mid |z-b| < r\}$. We say that D is completely regular if for any such closed ball C, all but a finite number of such open balls C_b are contained in D. It is easy to see that (1) this property is preserved under the action of the linear fractional group, (2) it is also preserved by a finite non-empty intersection and by a union of a chain, and (3) a connected open affine subset of $P^1(k)$ is completely regular. Furthermore we obtained in [5]

THEOREM 1. Let k be maximally complete, D a completely regular quasi-connected set $\subsetneqq P^1(k)$. Then there exists a sequence $\{D_n\}_{n=1}^{\infty}$ of connected open affine subsets of $P^1(k)$ such that $D_1 \subseteqq D_2 \subseteqq \cdots \subseteqq D_n \subseteqq \cdots$ and $D = \bigcup_{n=1}^{\infty} D_n$. Furthermore any two such sequences are cofinal.

§2. Relation between completely regular quasi-connected sets and rigid analytic spaces

As in §1, Krasner used Runge's theorem to construct his theory of analytic functions. On the other hand, Tate constructed his theory by adding one structure on analytic spaces:

Let k be as in §1. Let $k\{t_1,\ldots,t_n\}$ be the ring consisting of all power series $\sum_{m_1,\ldots,m_n \geqq 0} c_{m_1\cdots m_n} t_1^{m_1} t_2^{m_2} \cdots t_n^{m_n}$ such that $c_{m_1\cdots m_n} \to 0$ for $m_1 + \cdots + m_n \to \infty$. We say that a k-algebra A is tft over k if A is isomorphic to a quotient of $k\{t_1,\ldots,t_n\}$. Let $\mathcal{A}$ be the category of all such k-algebras. For any $A \in \mathcal{A}$, let Max(A) be the set of maximal ideals of A. Then Max(A) has a natural structure of an analytic space over k, and any analytic space over k is locally isomorphic to some Max(A) ($A \in \mathcal{A}$).

Let X be an analytic space over k. Then an h-structure on X is a selection, for each $A \in \mathcal{A}$, of a certain subset of k-ringed space morphisms $\mathrm{Max}(A) \to X$ satisfying two conditions (h1) and (h2). An h-space over k is an analytic space over k with an h-structure, and a rigid analytic space over k is an h-space over k which has a "good" affine covering. (For more details, see Tate [8].)

Now we are going to study the relation between these two theories. We quote the following result from [6].

THEOREM 2. Let k be as in §1, A a tft over k. We assume that A is an integral domain. Then for any element f of A, the image of

$$\mathrm{Max}(A) \ni p \longmapsto f(p) \in k$$

is a point or a connected open affine subset of $P^1(k)$.

Let D be a connected open affine subset of $P^1(k)$. Then $\mathcal{O}_0(D)$ is tft over k and the space $\mathrm{Max}(\mathcal{O}_0(D))$ is canonically isomorphic to D. Hence D has a natural structure of a rigid analytic space, and $\mathcal{O}_0(D)$ can be identified with the set of all h-morphisms of D to k (cf. Tate [8]).

Let A be tft over k. We say that a map f of Max(A) to $P^1(k)$ is structural if there exists a connected open affine subset D of $P^1(k)$ such that f is a composite of an h-morphism of Max(A) to D and the inclusion map $D \hookrightarrow P^1(k)$. Then this defines a structure of a rigid analytic space on $P^1(k)$ (cf. Tate [8], Proposition 10.15).

Now we assume that k is *maximally complete*. Let D be a completely regular quasi-connected set $\subsetneqq P^1(k)$, $\{D_n\}_{n=1}^{\infty}$ as in Theorem 1. Since D is an open subset of the h-space $P^1(k)$, D has an h-structure (cf. Tate [8], p.282). Let A be tft over k, and let f be an h-morphism of Max(A) to D. Then it follows Theorem 2 and the definition of the h-structure of D that the image of f is a finite union of points and connected open affine subsets contained in D. It follows from Proposition 2.5 of [5] that the image of f is contained in D_n for any sufficiently large n. This shows that $\{D_n\}_{n=1}^{\infty}$ is an admissible covering of D and the h-structure of D can be obtained by gluing up the natural h-structures of the D_n's (cf. Tate [8], Proposition 10.15). Hence D is a quasi-Stein space of Kiehl [3]. Furthermore a map f of D to k is an h-morphism iff the restriction of f to D_n is always an h-morphism (cf. Tate [8], Proposition 10.14). Hence f is an h-morphism iff the restriction of f to D_n is always an analytic function of Krasner (cf. Example 1). Therefore a map of D to k is an h-morphism iff f is an analytic function in the sense of Krasner.

Let $\mathcal{D} = \{D^{(m)}\}_{m=1}^{\infty}$ be a chain of a countable number of completely regular quasi-connected sets satisfying $\bigcup_{m=1}^{\infty} D^{(m)} = D$. By renumbering, we may assume $D^{(M)} \cap \bigcup_{m=1}^{M-1} D^{(m)} \neq \emptyset$ for any $M \geqq 1$. Let $\{D_n^{(m)}\}_{n=1}^{\infty}$ be as in Theorem 1. Here we may assume that $D_M^{(M)} \cap \bigcup_{m=1}^{M-1} D_M^{(m)} \neq \emptyset$ for any $M \geqq 1$. Let $D'_M = \bigcup_{m=1}^{M} D_M^{(m)}$. Then $\{D'_n\}_{n=1}^{\infty}$ satisfies the condition of Theorem 1. Let $f : \mathrm{Max}(A) \to D$ be an h-morphism. Then the image of f is contained in D'_n for any sufficiently large n. Since $\bigcup_{n=1}^{\infty} \{D_n^{(m)}\}_{m=1}^{\infty}$ is a refinement of $\{D^{(m)}\}_{m=1}^{\infty}$, this shows that $\mathcal{D}$ is an admissible covering of D. Hence, by the result of Kiehl [4], $H^p(\mathcal{D},\mathcal{O}) = 0$ for any positive integer p.

Therefore we have proved

THEOREM 3. *Let k be maximally complete, D a completely regular quasi-connected set $\subsetneqq P^1(k)$, $\{D_n\}_{n=1}^{\infty}$ as in Theorem 1. Then*

(1) The natural h-structure of $P^1(k)$ induces on D a structure of a quasi-Stein space. More closely, this structure can be obtained by gluing up the natural h-structure of the D_n's.

(2) $\mathcal{O}(D)$ can be identified with the set of all h-morphisms of D to k.

(3) Let $\mathcal{D} = \{D^{(m)}\}_{m=1}^{\infty}$ be a chain of a countable number of completely regular quasi-connected sets satisfying $\bigcup_{m=1}^{\infty} D^{(m)} = D$. Then $\mathcal{D}$ is an admissible covering of D. In particular $H^p(\mathcal{D},\mathcal{O}) = 0$ for any positive integer p.

Now we assume that k is not maximally complete. Then there exists a decreasing sequence $C_1 \supseteqq C_2 \supseteqq \ldots \supseteqq C_n \supseteqq \ldots$ of open balls in k such that the diameter of each C_n belongs to $|k^{\times}|$ and $\bigcap_{n=1}^{\infty} C_n \neq \emptyset$. Then $D_n = P^1(k) \setminus D_n$ is a connected open affine subset of $P^1(k)$. Hence.

by gluing up the natural h-structures of the D_n's, we can construct on $P^1(k)$ a structure of a quasi-Stein space. Of course, this h-structure is different from the natural h-structure of $P^1(k)$.
It is easy to see that any h-morphism of this quasi-Stein space $P^1(k)$ to k is an analytic function in the sense of Krasner.

Let k be as above. Let D be a completely regular quasi-connected set, $D_{a,b}$ $(a,b \in D)$ as in [5], 1-4. Then $D_{a,b}$ is a connected open affine subset of $P^1(k)$ and $\mathcal{D} = \{D_{a,b}\}_{a,b\in D}$ is a chain. Furthermore, it follows from Theorem 2 that $\mathcal{D}$ is an admissible covering of D. Hence a mapping of D to k is an h-morphism iff the restriction of it to each $D_{a,b}$ is an h-morphism. Therefore, in view of the definition of the h-structure of $D_{a,b}$, such a mapping is an analytic function in the sense of Krasner.

Let k be as in §1 and let D be any open subset of $P^1(k)$. For any $x,y \in D$, we write $x \sim y$ if there exists a completely regular quasi-connected set sub. of D containing x and y. Then this $\sim$ is an equivalence relation. Let $D = \coprod_i D_i$ be the corresponding decomposition of D.
Then it follows from Theorem 2 that (1) each D_i is a maximal completely regular quasi-connected subset of D, (2) $\mathcal{D} = \{D_i\}$ is an admissible covering of the h-space D, and (3) a mapping f of D to k is an h-morphism iff the restriction of f to each D_i is an h-morphism.

EXAMPLE 2. Let k be maximally complete, $D = \{z \in k \mid 1/2 < |z| < 2, |z| \neq 1\}$. Then D is a quasi-connected set. Hence $\mathcal{O}(D)$ is an integral domain.
On the other hand, the set of all h-morphisms of D to k is the direct sum of $\mathcal{O}(\{z \in k \mid 1/2 < |z| < 1\})$ and $\mathcal{O}(\{z \in k \mid 1 < |z| < 2\})$.

The following example seems to suggest that the class of completely regular quasi-connected sets is more natural than the class of quasi-connected sets.

EXAMPLE 3. Let $D = \{z \in \mathbb{C} \mid 1/2 < |z| < 2, |z| \neq 1\}$. Let $\mathfrak{F}(D)$ be the set consisting of all functions f of D to $\mathbb{C}$ such that there exists a sequence $\{f_n\}_{n=1}^{\infty}$ of rational functions without any poles in D and satisfying $f = \lim_{n\to\infty} f_n$ uniformly on D. Then each element f of $\mathfrak{F}(D)$ is a sum of a rational function and an analytic function on $\{z \in C \mid 1/2 < |z| < 2\}$. Hence the theorem of identity holds for functions of $\mathfrak{F}(D)$.

REMARK. The main difference of Krasner's analytic functions and Tate's rigid analytic spaces seems to be in the following point: Let D be a completely regular quasi-connected set. Let f be a function of D to k. Then (1) f is an analytic function iff there exists a chain $\mathcal{D}$ of quasi-connected sets such that the restriction of f to each $D' \in \mathcal{D}$ is an analytic element. On the other hand, (2) f is an h-morphism of D to k iff the restriction of f to each $D_{a,b}$ $(a,b \in D)$ is an analytic element. It is obvious that (2) implies (1). If k is maximally complete, then (1) implies (2). But (2) is stronger than (1) in general.

REMARK. The author proved Theorem 2 and Theorem 3 in March, 1978. He stated it at the symposium as an open problem.

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FINITEDIMENSIONAL NORMED K-VECTORSPACES

by

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§0. Introduction

The field K is supposed to be complete with respect to a non-Archimedean valuation. A norm $|| \ ||$ on a finitedimensional vector-space V over k is called <u>split</u> if V has a base $e_1,\dots,e_n$ such that

$$||\Sigma \lambda_i e_i|| = \max_i |\lambda_i| \, ||e_i|| \text{ for all } \lambda_1,\dots,\lambda_n \in k.$$

The normed vectorspace $(V, || \ ||)$ is also called <u>split</u> and $\{e_1,\dots,e_n\}$ is called an orthogonal base of V. It is well known (For Banachspaces over k the reader can consult [2], [4] and [5]) that for a maximally complete field k every norm on a finitedimensional vectorspace is split. So our main interest lies in the case where the base field k is not maximally complete. The work in this paper is centered around the following questions:

Q_0. What are the norms on a finitedimensional vectorspace?

Q1. What group of isometries does a finitedimensional normed vector-space have?

This setup is of course related to K-theory where K_0R describes the projective modules over R and K_1R describes the automorphism groups of projective modules. We will use very little K-theory (the reader may consult [1] or [3]). In §1 some K_0's are discussed and finally in §1.4 finitedimensional normed spaces over k are classified by (non-trivial) linear algebra over $\overline{k}$ = the residue field of k. In §2 the group of isometries of a normed finitedimensional vectorspace