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ON THE CONTINUOUS DEPENDENCE OF THE POLES OF THE SCATTERING MATRIX ON THE COEFFICIENTS OF AN ELLIPTIC OPERATOR

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ABSTRACT. The behavior of the poles of the analytic continuation to the nonphysical sheet of the resolvent of the operator $l_\varepsilon = -\nabla \cdot A_\varepsilon(x)\nabla$ is considered as $\varepsilon \to 0$, where $A_\varepsilon(x) = A_0(x) + \varepsilon a(x)$ is a smooth, positive-definite, matrix-valued function. It is proved that as $\varepsilon \to 0$ the poles of the kernel of the resolvent $(l_\varepsilon - z^2)^{-1}$ within an arbitrary compact set W lie only in the union of disks $\{z: |z - z_n| = O(\varepsilon^{1/q_n})\}$, $n = 1, \ldots, N(W)$, where q_n is the order of the pole z_n^2 of the kernel of the resolvent $(l_0 - z^2)^{-1}$.

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1. We consider the equation

$$u_{tt}(x,t) = \nabla \cdot A_\varepsilon(x)\nabla u(x,t), \qquad x \in \mathbb{R}^3,$$
$$u(x,0) = f_1(x), \qquad u_t(x,0) = f_2(x), \tag{1}$$

describing the process of scattering of acoustic waves by inhomogeneities of the medium. Here the matrix-valued function $A_\varepsilon(x) = A_0(x) + \varepsilon a(x)$ is assumed to be positive definite, $\xi \cdot A_\varepsilon(x)\xi \geqslant c_1|\xi|^2$, $c_1 > 0$, and smooth (for example, $A_\varepsilon(x) \in C^1$) for any value of the numerical parameter $\varepsilon \in [0,1]$. Moreover, $A_\varepsilon(x) = A_0(x) = I$, $|x| > \rho$, where I is the identity matrix.

It is known [1] that the scattering matrix $S_\varepsilon(z)$ of problem (1) is holomorphic in the lower half-plane $\operatorname{Im} z \leqslant 0$ and meromorphic in the upper half-plane $\operatorname{Im} z > 0$. The poles of the scattering matrix $S_\varepsilon(z)$ coincide with the poles of the analytic continuation to the nonphysical sheet ($\operatorname{Im} z > 0$) of the kernel of the resolvent $(l_\varepsilon - z^2)^{-1}$ of the corresponding steady-state problem.

In this paper we show (Theorem 3) that as $\varepsilon \to 0$ the poles of the scattering matrix $S_\varepsilon(z)$ lying within an arbitrary compact set W lie only in the union of disks

$$\{z \mid |z - z_n| = O(\varepsilon^{1/q_n})\}, \qquad n = 1, \ldots, N(W),$$

where q_n is the order of the pole z_n^2 of the kernel of the resolvent $(l_0 - z^2)^{-1}$. The proof of this assertion is based on an operator interpretation of the nonphysical sheet. Namely, from the viewpoint of the geometry of Hilbert space, to the analytic continuation of the kernel of the resolvent $(l_\varepsilon - z^2)^{-1}$, $0 \leqslant \varepsilon \leqslant 1$, into the upper half-plane $\operatorname{Im} z > 0$ there corresponds its replacement, according to the scheme of Lax and Phillips [1], by the resolvent $(B_\varepsilon - z)^{-1}$ of a certain dissipative operator B_ε whose eigenvalues coincide with the poles of the scattering matrix $S_\varepsilon(z)$.

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2. Let H^ε, $0 \leqslant \varepsilon \leqslant 1$, be the closure of the set of infinitely differentiable, compactly supported functions $C_0^\infty(\mathbb{R}^3)$ in the norm

$$\|\varphi\|^2_{H^\varepsilon} = \int_{\mathbb{R}^3} \overline{\nabla\varphi} \cdot A_\varepsilon(x)\nabla\varphi\, dx$$

and let $L^2 = L^2(\mathbb{R}^3)$ be the space of square-summable functions. We denote by $\mathscr{H}^\varepsilon$ the energy space $H^\varepsilon \oplus L^2$ with norm

$$\|u\|^2_{\mathscr{H}^\varepsilon} = \tfrac{1}{2}\left(\|u_1\|^2_{H^\varepsilon} + \|u_2\|^2\right).$$

In the space $\mathscr{H}^\varepsilon$, $0 \leqslant \varepsilon \leqslant 1$, we consider the Cauchy problem for the wave equation (1)

$$\frac{\partial}{\partial t}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \nabla \cdot A_\varepsilon \nabla & 0 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \qquad \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}\bigg|_{t=0} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \tag{2}$$

setting $u(x,t) = u_1(x,t)$ and $u_t(x,t) = u_2(x,t)$.

We denote by iL_ε the matrix operator

$$A_\varepsilon = \begin{pmatrix} 0 & 1 \\ \nabla \cdot A_\varepsilon(x)\nabla & 0 \end{pmatrix}$$

on the domain $\mathscr{D}(L_\varepsilon) = \{u \in \mathscr{H}^\varepsilon\colon A_\varepsilon u \in \mathscr{H}^\varepsilon\}$, $0 \leqslant \varepsilon \leqslant 1$. In $\mathscr{H}^\varepsilon$ the skew-symmetric operator iL_ε generates [1] the unitary group $U_\varepsilon(t) = \exp\{iL_\varepsilon t\}$ of solution operators of problem (2).

We denote by $\mathscr{D}_+$ and $\mathscr{D}_-$ the outgoing and incoming subspaces of $\mathscr{H}^\varepsilon$ [1], i.e., the set of all Cauchy data to which there correspond solutions of (2) equal to zero in forward $\{|x| < \rho + t,\ t \geqslant 0\}$ and backward $\{|x| < \rho - t,\ t \leqslant 0\}$ frustums of cones respectively. We note that $\mathscr{D}_+$ and $\mathscr{D}_-$ are orthogonal in $\mathscr{H}^\varepsilon$, $0 \leqslant \varepsilon \leqslant 1$.

Let P_{K^ε} be orthogonal projection onto the subspace

$$K^\varepsilon = \mathscr{H}^\varepsilon \ominus \{\mathscr{D}_+ \oplus \mathscr{D}_-\}, \qquad 0 \leqslant \varepsilon \leqslant 1.$$

Then for any fixed $\varepsilon \in [0,1]$ the family of operators

$$\mathscr{Z}_\varepsilon(t) = P_{K^\varepsilon} U_\varepsilon(t) P_{K^\varepsilon}, \qquad t \geqslant 0,$$

is a strongly continuous semigroup of contractions in K^ε [1].

Let iB_ε be the generator of the semigroup $\mathscr{Z}_\varepsilon(t)$, $0 \leqslant \varepsilon \leqslant 1$. Then the resolvent $(B_\varepsilon - z)^{-1}$ of the dissipative operator B_ε is the projection onto K^ε of the resolvent $(L_\varepsilon - z)^{-1}$ of the selfadjoint operator L_ε, i.e.,

$$(B_\varepsilon - z)^{-1} = P_{K^\varepsilon}(L_\varepsilon - z)^{-1} P_{K^\varepsilon}, \qquad \operatorname{Im} z < 0. \tag{3}$$

By explicitly inverting the matrix operator $L_\varepsilon - z$, it is easy to obtain

$$(L_\varepsilon - z)^{-1} = \begin{pmatrix} zG_\varepsilon(z^2) & -iG_\varepsilon(z^2) \\ i\left[z^2 G_\varepsilon(z^2) + I\right] & zG_\varepsilon(z^2) \end{pmatrix}, \qquad \operatorname{Im} z \neq 0, \tag{4}$$

where $G_\varepsilon(z^2)$ is an integral operator with kernel which is the Green function $G_\varepsilon(x, y; z^2)$ of the differential operator $-\nabla \cdot A_\varepsilon(x)\nabla - z^2$.

We wish to show that in an appropriate sense

$$\lim_{\varepsilon \to 0} (B_\varepsilon - z)^{-1} = (B_0 - z)^{-1}.$$

Since, according to (3), the resolvent $(B_\varepsilon - z)^{-1}$ of the dissipative operator B_ε is equal to the projection onto the subspace K^ε of the resolvent $(L_\varepsilon - z)^{-1}$ of the selfadjoint operator L_ε, $0 \leqslant \varepsilon \leqslant 1$, we first prove the following assertion.

THEOREM 1. *There exists a constant C such that*

$$\|(L_\varepsilon + i)^{-1} - (L_0 + i)^{-1}\|_{\mathscr{H}^0} \leqslant C\varepsilon.$$

PROOF. We observe that because of (4) the theorem follows from the assertions

$$\|G_\varepsilon(-1) - G_0(-1)\|_{L^2 \to L^2} = O(\varepsilon), \tag{5}$$

$$\|G_\varepsilon(-1) - G_0(-1)\|_{L^2 \to H^0} = O(\varepsilon), \tag{6}$$

$$\|G_\varepsilon(-1) - G_0(-1)\|_{H^0 \to L^2} = O(\varepsilon), \tag{7}$$

$$\|G_\varepsilon(-1) - G_0(-1)\|_{H^0 \to H^0} = O(\varepsilon). \tag{8}$$

We first prove (5) and (6). Let $u_\varepsilon = G_\varepsilon(-1)\varphi$, $\varphi \in C_0^\infty(\mathbb{R}^3)$, $\varepsilon \in [0,1]$, i.e., the function $u_\varepsilon(x)$ satisfies the differential equation

$$-\nabla \cdot A_\varepsilon \nabla u_\varepsilon + u_\varepsilon = \varphi, \qquad x \in \mathbb{R}^3.$$

Using the formula for integration by parts, we obtain

$$\int_{\mathbb{R}^3} \left(\nabla \bar{u}_\varepsilon \cdot A_\varepsilon(x)\nabla u_\varepsilon + |u_\varepsilon|^2\right) dx = \int_{\mathbb{R}^3} \varphi \bar{u}_\varepsilon\, dx,$$

from which it follows that

$$\|u_\varepsilon\|^2_{H^\varepsilon} + \|u_\varepsilon\|^2 \leqslant \|\varphi\|^2, \qquad \varepsilon \in [0,1]. \tag{9}$$

Suppose further that $v_\varepsilon = u_\varepsilon - u_0$. The function $v_\varepsilon(x)$ is a solution of the differential equation

$$-\nabla \cdot A_0(x)\nabla v_\varepsilon + v_\varepsilon = \varepsilon \nabla \cdot a(x)\nabla u_\varepsilon, \qquad x \in \mathbb{R}^3.$$

Multiplying this equation by $\bar{v}_\varepsilon(x)$ and then integrating by parts, we obtain

$$\int_{\mathbb{R}^3} \left(\overline{\nabla v_\varepsilon} \cdot A_0(x)\nabla v_\varepsilon + |v_\varepsilon|^2\right) dx = -\varepsilon \int_{|x|<\rho} \overline{\nabla v_\varepsilon} \cdot a(x)\nabla u_\varepsilon\, dx. \tag{10}$$

Since

$$\left|\int_{|x|<\rho} \overline{\nabla v_\varepsilon} \cdot a(x)\nabla u_\varepsilon\, dx\right| \leqslant c_2 \left(\int_{|x|<\rho} |\nabla v_\varepsilon|^2\, dx\right)^{1/2} \left(\int_{|x|<\rho} |\nabla u_\varepsilon|^2\, dx\right)^{1/2}, \tag{11}$$

where $c_2 = \sup_{|x|<\rho} \|a(x)\|$, and

$$c_1 \|\nabla u_\varepsilon\| \leqslant \|u_\varepsilon\|_{H_\varepsilon}; \qquad c_1 \|\nabla v_\varepsilon\| \leqslant \|v_\varepsilon\|_{H^0}, \tag{12}$$

from (10), with (9) taken into account, we obtain

$$\left(\|v_\varepsilon\|^2_{H^0} + \|v_\varepsilon\|^2\right)^{1/2} \leqslant c_1^{-2} c_2 \varepsilon \|\varphi\|,$$

which obviously proves (5) and (6).

To prove (7) and (8) it suffices to show that for some constant c_3

$$\left(\int_{|x|<\rho} |\nabla u_\varepsilon|^2\, dx\right)^{1/2} \leqslant c_3 \|p\varphi\|, \tag{13}$$

where $p(x)$ is an infinitely differentiable function satisfying the conditions: 1) $p(x) = 1, |x| < \rho$; 2) $(\rho + 1)^{-1} \leqslant p(x) \leqslant 1$, $\rho < |x| < \rho + 1$; 3) $p(x) = |x|^{-1}$, $|x| \geqslant \rho + 1$.

Indeed, from (10) and (11)–(13) we obtain

$$\left(\|v_\varepsilon\|^2_{H^0} + \|v_\varepsilon\|^2\right)^{1/2} \leqslant c_1^{-1} c_2 c_3 \varepsilon \|p\varphi\|.$$

On the other hand (see [2], Chapter VI, §4.3),

$$\|p\varphi\| \leqslant c_4 \|\nabla\varphi\|.$$

We thus see that

$$\left(\|v_\varepsilon\|^2_{H^0} + \|v_\varepsilon\|^2\right)^{1/2} \leqslant c_1^{-2} c_2 c_3 c_4 \varepsilon \|\varphi\|_{H^0}$$

from which (7) and (8) obviously follow.

It thus remains for us to prove (13). For this we note that the Green function $G_\varepsilon(x, y; -1)$, $\varepsilon \in [0, 1]$, satisfies the conditions [3]

$$|\nabla_x G_\varepsilon(x, y; -1)| = O\left(|x - y|^{-2}\right), \qquad |x| < \rho, |y| < \rho,$$

$$|\nabla_x G_\varepsilon(x, y; -1)| = O\left(|y|^{-1} \exp\{-|y|\}\right), \qquad |x| < \rho, |y| > \rho + 1,$$

from which it follows that the integral operator generated by the function

$$|\nabla_x G_\varepsilon(x, y; -1)| p(y)^{-1}$$

is bounded from $L^2(\mathbb{R}^3)$ to $L^2(\{|x| < \rho\})$, i.e., (13) holds.

THEOREM 2. *The family of resolvents $(B_\varepsilon - z)^{-1}$ of the dissipative operators B_ε converges uniformly as $\varepsilon \to 0$ at the regular point $z = -i$ to the resolvent $(B_0 - z)^{-1}$ of the dissipative operator B_0, and, moveover,*

$$\|(B_\varepsilon + i)^{-1} - (B_0 + i)^{-1}\|_{\mathscr{H}^0} \leqslant C\varepsilon.$$

Indeed, since the restriction of any element $u \in \mathscr{D}_+ \oplus \mathscr{D}_-$ to the set $\{x: |x| < \rho\}$ is equal to zero, all the subspaces K^ε, $0 \leqslant \varepsilon \leqslant 1$, coincide topologically. Therefore, the resolvent $(B_\varepsilon + i)^{-1}$ considered in the subspace K^0 has the form

$$(B_\varepsilon + i)^{-1} u = P_{K^0}(L_\varepsilon + i)^{-1} u, \qquad u \in K^0.$$

Hence,

$$\|(B_\varepsilon + i)^{-1} - (B_0 + i)^{-1}\|_{\mathscr{H}^0} \leqslant \|(L_\varepsilon + i)^{-1} - (L_0 + i)^{-1}\|_{\mathscr{H}^0},$$

from which Theorem 2 follows by Theorem 1.

3. We now formulate our main assertion. Let W be a compact set in the complex z-plane containing in its interior part of the spectrum of the operator B_0. Further, let q_n be the order of a pole z_n, $n = 1, \ldots, N$, of the resolvent $(B_0 - z)^{-1}$ inside the compact set W, and let $C(r_n) = \{z: |z - z_n| \leqslant r_n\}$. Then the following theorem holds.

THEOREM 3. *There exist positive constants $\Delta = \Delta(W)$, $M_1 = M_1(W)$, and $M_2 = M_2(W)$ such that for any $\varepsilon \leqslant \Delta$ the following assertions are true:*

1) The intersection of the spectrum of the operator B_ε with the compact set W lies in the union of the disks $C(r_n)$, $n = 1, \ldots, N$, where $r_n = M_1 \varepsilon^{1/q_n}$.

2) For the spectral spaces $P_\varepsilon^{(n)}$ and $P_0^{(n)}$ corresponding to the part of the spectrum of the operators B_ε and B_0 lying in the disk $C(r_n)$ there is the estimate

$$\|P_\varepsilon^{(n)} - P_0^{(n)}\|_{\mathscr{H}^0} \leqslant M_2 \varepsilon.$$

The proof (see [4]) is based on Theorem 2, the inequality (see [2], Chapter IV, §3.3)

$$\|(B_\varepsilon - z)^{-1} - (B_0 - z)^{-1}\| \leqslant \frac{\|1 + (z+i)(B_0 - z)^{-1}\| \, \|(B_\varepsilon + i)^{-1} - (B_0 + i)^{-1}\|}{1 - |z + i| \, \|1 + (z+i)(B_0 - z)^{-1}\| \, \|(B_\varepsilon + i)^{-1} - (B_0 + i)^{-1}\|},$$

which holds under the condition that the denominator on the right side is positive, and on the well-known integral representation of spectral projections

$$P_\varepsilon^{(n)} - P_0^{(n)} = -\frac{1}{2\pi i} \int_{\partial C(r_n(\Delta))} \left[(B_\varepsilon - z)^{-1} - (B_0 - z)^{-1}\right] dz.$$

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