BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sciences math., astr.
et phys. — Vol. XV, No. 3, 1967

MATHEMATICS
(CALCULUS OF PROBABILITY)

A Random Integral and Orlicz Spaces

by

K. URBANIK and W. A. WOYCZYŃSKI

Presented by K. URBANIK on December 2, 1966

1. Symmetric random measures. Let $\mathcal{B}$ be the σ-algebra of all Borel subsets of the unit interval $I = [0, 1]$. A function M defined on $\mathcal{B}$ whose values are real-valued random variables is called a *random measure* if

(i) for every sequence $E_1, E_2, \ldots$ of disjoint sets from $\mathcal{B}$ $M(\bigcup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} M(E_n)$, where the series converges with probability 1,

(ii) for every sequence E_1 E_2 ... E_n of disjoint sets from $\mathcal{B}$ the random variables $M(E_1), M(E_2), \ldots, M(E_n)$ are independent.

For the theory of random measures see [9]—[11].

A random measure M is said to be *symmetric* if the random variables $M(E)$ $(E \in \mathcal{B})$ are symmetrically distributed. Further, a random measure M is said to be *atomless* if $M(\{a\}) = 0$ for every one-point set $\{a\}$. In this paper we identify random variables which are equal with probability 1. If M is an atomless random measure, then for every $E \in \mathcal{B}$ the random variable $M(E)$ has an infinitely divisible distribution ([11], p. 380). Thus the characteristic function $\varphi_{M(E)}$ of the random variable $M(E)$ can be written in the Lévy—Khinchine form

$$(1.1) \qquad \varphi_{M(E)}(t) = \exp\left[i\gamma_M(E)\, t + \int_{-\infty}^{\infty} \left(e^{itu} - 1 - \frac{itu}{1+u^2}\right)\frac{1+u^2}{u^2}\, dH_E(u)\right],$$

where the function H_E is monotone non-decreasing, bounded, and $H_E(-\infty) = 0$. The set functions $\gamma_M(E)$ and $\mu_M(E) = H_E(\infty)$ $(E \in \mathcal{B})$ are number-valued Borel measures ([11], p. 381). Moreover, $M(E) = 0$ if and only if $\gamma_M(E) = \mu_M(E) = 0$. A set E from $\mathcal{B}$ is said to be an M-null set if $M(A) = 0$ for all Borel subsets A of E. Relations valid except on an M-null set are said to be valid M-almost everywhere. Put $\nu_M = \operatorname{var} \gamma_M + \mu_M$. Then the classes of M-null sets and ν_M-null sets are identical. Now we shall prove an analogue of the Nikodym Theorem. The limit in probability of a sequence X_n of random variables will be denoted by $\mathrm{p} - \lim_{n\to\infty} X_n$.

THEOREM 1.1. *Let $M_1, M_2, \ldots$ be a sequence of atomless random measures. Suppose that for every $E \in \mathcal{B}$ there exists the limit*

$$\text{p} - \lim_{n\to\infty} M_n(E) = M(E). \tag{1.2}$$

Then M is an atomless random measure.

Proof. For every $E \in \mathcal{B}$ the convergence (1.2) implies the uniform convergence, $\lim_{n\to\infty} \varphi_{M_n(E)}(t) = \varphi_{M(E)}(t)$ in every finite interval. Thus $M(E)$ has an infinitely divisible distribution and its characteristic function is given by (1.1), where $\lim_{n\to\infty} \gamma_{M_n}(E) = \gamma_M(E)$ and $\lim_{n\to\infty} \mu_{M_n}(E) = \mu_M(E)$ ([5], p. 300). By the Nikodym Theorem ([3], Chap. III, 7) both set functions γ_M and μ_M are atomless measures. Further, it is clear that the function M is atomless, finitely additive and satisfies condition (ii). Suppose that $E_1 \supset E_2 \supset \ldots$ and $\bigcap_{n=1}^{\infty} E_n = \emptyset$, $(E_n \in \mathcal{B}, n = 1, 2, \ldots)$. Then $\lim_{n\to\infty} \gamma_M(E_n) = \lim_{n\to\infty} \mu_M(E_n) = 0$. Consequently, by (1.1) $\lim_{n\to\infty} \varphi_{M(E_n)}(t) = 1$ uniformly in every finite interval. Thus $\text{p} - \lim_{n\to\infty} M(E_n) = 0$ which, by Prékopa's Theorem ([9], p. 227) shows that M is completely additive. Consequently M is an atomless random measure.

2. A random integral. Let M be an atomless random measure. If f is a real valued Borel simple function, $f = \sum_{j=1}^{n} c_j \chi_{E_j}$, where $E_1, E_2, \ldots, E_n$ belong to $\mathcal{B}$ and χ_A denotes the indicator of A, then the integral on every Borel set E of f with respect to M is defined by the formula

$$\int_E f(s)\, M(ds) = \sum_{j=1}^{n} c_j M(E_j \cap E).$$

It is clear that this definition does not depend upon a particular representation of f in the form of a linear combinations of indicators. Further, the integral of every Borel simple function is a random measure. A Borel function f defined on I is said to be *M-integrable* if there exists a sequence of Borel simple functions $\{f_n\}$ such that

(*) the sequence $\{f_n\}$ converges to f M-almost everywhere on I, .

(**) for every $E \in \mathcal{B}$ the sequence $\{\int_E f_n(s)\, M(ds)\}$ converges in probability.

Then we put $\int_E f(s)\, M(ds) = \text{p} - \lim_{n\to\infty} \int_E f_n(s)\, M(ds)$.

Now we shall prove that the *integral of an M-integrable function is uniquely determined*. Let $\{f_n\}$ and $\{g_n\}$ be two sequences of Borel simple functions satisfying conditions (*) and (**). We have to prove that the corresponding sequences of integrals have the same limits. Put $h_n = f_n - g_n$ $(n = 1, 2, \ldots)$. The sequence $\{h_n\}$ tends to zero M-almost everywhere on I and consequently, ν_M-almost everywhere,

where ν_M is a non-negative measure associated with M. Thus, by the Egorov Theorem ([4], sec. 2.1) every set E from $\mathcal{B}$ is a union

$$E = \bigcup_{k=0}^{\infty} E_k \tag{2.1}$$

of disjoint sets from $\mathcal{B}$ such that $\nu_M(E_0) = 0$ and $\lim_{n\to\infty} h_n = 0$ uniformly on every set $E_1, E_2, \ldots$. Let γ_{N_n} and μ_{N_n} be number valued measures corresponding to the random measure $N_n(A) = \int_A h_n(s)\, M(ds)$. Since $|\gamma_{N_n}(A)| \leqslant \operatorname{Var} \gamma_M(A) \sup_{s\in A} |h_n(s)|$ and $\mu_{N_n}(A) \leqslant \mu_M(A) \sup_{s\in A} |h_n(s)|$, we have the formulae $\lim_{n\to\infty} \gamma_{N_n}(E_k) = \lim_{n\to\infty} \mu_{N_n}(E_k) = 0$, $(k = 0, 1, \ldots)$. Thus

$$\text{p} - \lim_{n\to\infty} N_n(E_k) = 0 \quad (k = 0, 1, \ldots). \tag{2.2}$$

Put $N(A) = \text{p} - \lim_{n\to\infty} N_n(A)$ $(A \in \mathcal{B})$. By Theorem 1.1, N is a random measure. Hence and from (2.1) and (2.2) we get the formula $N(E) = \sum_{k=0}^{\infty} N(E_k) = 0$ for every $E \in \mathcal{B}$. Thus for every set $E \in \mathcal{B}$ the sequences $\{\int_E f_n(s)\, M(ds)\}$ and $\{\int_E g_n(s)\, M(ds)\}$ have the same limit.

The above definition of the random integral is an adaptation of the Dunford's definition of the integral with respect to a measure whose values belong to a Banach space ([3], Chap. IV, 10). One can prove that our definition of the integral is equivalent to the Prékopa's definition of the unconditional integral ([10], p. 340). It is evident that the random integral is a linear operation. Moreover, if the Borel sets $E_1, E_2, \ldots, E_n$ are disjoint, then the random variables $\int_{E_1} f(s)\, M(ds), \int_{E_2} f(s)\, M(ds), \ldots, \int_{E_n} f(s)\, M(ds)$ are independent.

Let M be a symmetric atomless random measure. Then the characteristic function $\varphi_{M(E)}$ of $M(E)$ $(E \in \mathcal{B})$ is of the form

$$\varphi_{M(E)}(t) = \exp \int_0^{\infty} (\cos tu - 1) \frac{1+u^2}{u^2}\, dG_E(u), \tag{2.3}$$

where the function G_E is monotone non-decreasing, bounded, continuous on the left, and normalized by the condition $G_E(0) = 0$. For the measure μ_M corresponding to M the equation $\mu_M(E) = G_E(\infty)$ $(E \in \mathcal{B})$ holds. Consequently, M and μ_M have the same class of null-sets.

LEMMA 2.1. *Let M be a symmetric atomless random measure. If $\{f_n\}$ is a sequence of Borel simple functions and $\text{p} - \lim_{n\to\infty} \int_I f_n(s)\, M(ds) = 0$, then $\{f_n\}$ converges to 0 in measure μ_M.*

Proof. Put $f_n = \sum_{j=1}^{k_j} c_{j,n} \chi_{E_{j,n}}$ $(n = 1, 2, \ldots)$, where $E_{1,n}, E_{2,n}, \ldots, E_{k_n,n}$ are disjoint Borel sets. Then the characteristic function ψ_n of $\int_I f_n(s)\, M(ds)$ is given by the formula $\psi_n(t) = \prod_{j=1}^{k_n} \varphi_{M(E_{j,n})}(c_{j,n} t)$ $(n = 1, 2, \ldots)$. Moreover, $\lim_{n\to\infty} \psi_n(t) = 1$ uniformly in every finite interval. Hence, by (2.3) for every $\varepsilon > 0$ we get the formula

$$\lim_{n\to\infty} \sum_{j=1}^{k_n} \int_0^1 \int_0^\infty (1-\cos c_{j,n} tu) \frac{1+u^2}{u^2} dG_{E_{j,n}\cap A_n}(u)\, dt = 0, \tag{2.4}$$

where $A_n = \{s : |f_n(s)| \geq \varepsilon\}$. Since

$$\int_0^1 (1-\cos c_{j,n} tu)\, dt \geq C \frac{|c_{j,n}|^2 u^2}{1+u^2} \geq C\varepsilon^2 \frac{u^2}{1+u^2} \quad \text{if} \quad E_{j,n} \cap A_n \neq \emptyset,$$

where C is a positive constant, we infer, in view of (2.4), that $\mu_M(A_n) = \sum_{j=1}^{k_n} G_{E_{j,n}\cap A_n}(\infty)$ tends to 0 if $n\to\infty$. In other words the sequence $\{f_n\}$ converges to 0 in measure μ_M.

LEMMA 2.2. *Let M be a symmetric atomless random measure. If $\{f_n\}$ is a sequence of Borel simple functions and the sequence of integrals $\{\int_I f_n(s) M(ds)\}$ converges in probability, then $\{f_n\}$ converges in measure μ_M to an M-integrable function f and for every $E \in \mathcal{B}$ $\mathrm{p}-\lim_{n\to\infty} \int_E f_n(s) M(ds) = \int_E f(s) M(ds)$.*

Proof. For every pair $\{n_k\}$ and $\{m_k\}$ of subsequences of the sequence of positive integers we have the formula

$$\mathrm{p}-\lim_{k\to\infty} \int_I (f_{n_k}(s) - f_{m_k}(s) M(ds)) = 0. \tag{2.5}$$

Thus, by Lemma 2.1, $\lim_{k\to\infty} (f_{n_k} - f_{m_k}) = 0$ in measure μ_M. Hence, it follows that $\{f_n\}$ is a Cauchy sequence with respect to the convergence in measure μ_M. Consequently, it converges to a Borel function f in measure μ_M. Since for every $E \in \mathcal{B}$ the random variables $\int_E (f_{n_k}(s) - f_{m_k}(s)) M(ds)$ and $\int_{I\setminus E} (f_{n_k}(s) - f_{m_k}(s)) M(ds)$ are independent and symmetrically distributed, we have by (2.5) the formula

$$\mathrm{p}-\lim_{k\to\infty} \int_E (f_{n_k}(s) - f_{m_k}(s)) M(ds) = 0.$$

Thus the sequence of random variables $\{\int_E f_n(s) M(ds)\}$ is a Cauchy sequence with respect to the convergence in probability. Consequently, it converges to a random variable $N(E)$. Taking a subsequence of $\{f_n\}$ convergent to f μ_M-almost everywhere and, consequently, M-almost everywhere we infer, by the definition of the random integral, that the function f is M-integrable and $\int_E f(s) M(ds) = N(E)$ $(E \in \mathcal{B})$. The Lemma is thus proved.

Let $\mathcal{L}(M)$ be the set of all M-integrable functions, where M is a symmetric atomless random measure. Of course, $\mathcal{L}(M)$ is a linear space under usual addition and scalar multiplication. We identify functions which are equal M-almost everywhere. From Lemma 2.1 it follows that $f = 0$ M-almost everywhere if and only if $\int_I f(s) M(ds) = 0$. Thus we may define a non-homogeneous norm in $\mathcal{L}(M)$ by

means of the formula $\|f\| = |||\int_I f(s) M(ds)|||$, where $|||X|||$ is the Fréchet norm of the random variable X i.e. the expectation $E\frac{|X|}{1+|X|}$. It should be noted that the convergence in Fréchet norm is equivalent to the convergence in probability. The space $\mathcal{L}(M)$ is a linear metric space under the norm $\|\ \|$. Moreover, from Lemma 2.2 it follows that the space $\mathcal{L}(M)$ is complete. The set of all Borel simple functions is dense in $\mathcal{L}(M)$.

3. Orlicz spaces. Let F_1 and F_2 be two non negative functions defined on the right half-line. We say that F_1 is *non-weaker* than F_2 and write $F_2 \prec F_1$ if $F_2(x) \leq aF_1(kx)$ for $x \geq x_0$ holds with some constants $a, k > 0$ and $x_0 \geq 0$. We say that F_1 and F_2 are *equivalent* and write $F_1 \sim F_2$ if $F_1 \prec F_2$ and $F_2 \prec F_1$. Let K be the class of all non-decreasing continuous functions defined on the right half-line vanishing only at the origin. Given $\Phi \in K$ for every Borel function f on I we put $R_\Phi(f) = \int_I \Phi(|f(s)|)\, ds$. Let $L(\Phi)$ be the set of all real valued Borel functions f on I such that $R_\Phi(cf)$ is finite for a positive constant c (in general dependent on f). The set $L(\Phi)$ is a linear space under usual addition and scalar multiplication. Moreover, it becomes a complete linear metric space under the non-homogeneous norm

$$\|f\|_\Phi = \inf\{c : c > 0, R_\Phi(c^{-1}f) \leq c\}.$$

The space $L(\Phi)$ with this norm is called an *Orlicz space*, [6] - [8].

In this paper the linear metric spaces $(X, \|\ \|_1)$ and $(X, \|\ \|_2)$ will be treated as identical if the convergence in the norm $\|\ \|_1$ is equivalent to the convergence in the norm $\|\ \|_2$. For two functions Φ and Ψ from K the equation $L(\Phi) = L(\Psi)$ holds if and only if $\Phi \sim \Psi$ ([6]).

We say that a function Φ satisfies the *Δ_2-condition* if $\Phi(2x) \leq b\Phi(x)$ for $x \geq x_0$ holds with some constants $b > 0$ and $x_0 \geq 0$. The Δ_2-condition for Φ is equivalent to the statement that the set of all Borel simple functions is dense in $L(\Phi)$.

Let K_0 be the class of all functions Φ from K for which $\frac{\Phi(y)}{y^2} \leq c\frac{\Phi(x)}{x^2}$, $y \geq x \geq x_0$ holds with some constants $c > 0$ and $x_0 \geq 0$. It is clear that all functions in K_0 satisfy the Δ_2-condition, and $\overline{\lim_{x\to\infty}} \frac{\Phi(x)}{x^2} < \infty$. Moreover, if $\Phi \in K_0$ and $\Phi \sim \Psi$ then $\Psi \in K_0$. It was proved in [6] and [7] (p. 109) that $\Phi \in K_0$ *if and only if $\Phi \sim \Psi$ and the function $\Psi(\sqrt{x})$ is concave.* As examples of functions belonging to K_0 we quote the functions Φ_1 with $\Phi_1(\infty) < \infty$, $\Phi_2(x) = x^p$ $(0 < p \leq 2)$, $\Phi_3(x) = x^q (\log x)^{r_1} (\log\log x)^{r_2} \cdot \ldots \cdot (\log\log\ldots\log x)^{r_n}$ for x sufficiently large $(0 < q < 2, r_1, r_2, \ldots, r_n \geq 0)$.

4. Homogeneous random measures. Throughout this paragraph we assume that the measure M in question is not identically equal to 0. A random measure M is said to be *homogeneous* if for each pair E_1, E_2 of congruent Borel sets the random variables $M(E_1)$ and $M(E_2)$ are identically distributed. Of course, homogeneous random measures are atomless. Moreover, for symmetric homogeneous measures M the characteristic function $\varphi_{M(E)}$ of $M(E)$ $(E \in \mathcal{B})$ is given by the formula (2.3) with

$$G_E(u) = |E|\, G(u), \tag{4.1}$$

where $|E|$ is the Lebesgue measure of E, and the Lévy–Khinchine function G is monotone non-decreasing, bounded, continuous on the left with $G(0) = 0$ and $G(\infty) > 0$. Hence, it follows that $M(E) = 0$ if and only if $|E| = 0$. Moreover, the measure μ_M corresponding to M is equal to the Lebesgue measure up to a positive factor.

We note that for every monotone non-decreasing bounded function G continuous on the left with $G(0) = 0$ and $G(\infty) > 0$ there exists a symmetric homogeneous random measure for which (4.1) holds. In fact, for every such function G there exists a separable homogeneous stochastic process $X(t)$ $(0 \leqslant t \leqslant 1)$ with independent increments such that the characteristic function $\psi_{[a,b)}$ of the increment $X(b) - X(a)$ is given by the formula

$$\psi_{[a,b)}(t) = \exp (b-a) \int_0^\infty (\cos tu - 1) \frac{1+u^2}{u^2}\, dG(u)$$

([1]; [2], p. 61, 605). Setting $M\left(\bigcup_{j=1}^{n} [a_j, b_j)\right) = \sum_{j=1}^{n} (X(b_j) - X(a_j))$ $(0 \leqslant a_1 < b_1 \leqslant \leqslant a_2 < b_2 \leqslant \ldots \leqslant a_n < b_n \leqslant 1)$ we get a random set function which by Prékopa's Theorems ([9], pp. 227, 243) can be extended to a random measure. Of course, this measure is symmetric, homogeneous and satisfies condition (4.1).

Now we assume that M is a symmetric homogeneous random measure with the Lévy–Khinchine function G. The characteristic function ψ_f of the integral $\int_I f(s)\, M(ds)$ $(f \in \mathcal{L}(M))$ is then given by the formula

$$\psi_f(t) = \exp\left(-\int_I T_M(tf(s))\, ds\right), \tag{4.2}$$

where

$$T_M(x) = \int_0^\infty (1 - \cos xu) \frac{1+u^2}{u^2}\, dG(u). \tag{4.3}$$

Put

$$U_M(x) = \int_0^\infty \min(x^2, u^{-2})(1+u^2)\, dG(u), \tag{4.4}$$

$$\Psi_M(x) = \int_{1/x}^\infty \frac{G(u)}{u^3}\, du. \tag{4.5}$$

Both functions U_M and Ψ_M belong to the class K.

LEMMA 4.1. *For all $x \geqslant 0$ and $a \geqslant 0$ inequalities*

$$\max_{0 \leqslant v \leqslant ax} T_M(v) \leqslant c_1(a)\, U_M(x),$$

$$\int_0^1 T_M(xt)\, dt \geqslant c_2\, U_M(x)$$

hold with some positive $c_1(a)$ and c_2. Moreover, $U_M \sim \Psi_M$.

Proof. The inequalities in question are a consequence of the inequalities

$$\max_{0 \leqslant v \leqslant ax} (1 - \cos vu) \leqslant c_1(a) \min(x^2 u^2, 1), \quad (x, u \geqslant 0),$$

$$1 - \frac{\sin xu}{xu} \geqslant c_2 \min(x^2 u^2, 1), \quad (x, u \geqslant 0)$$

and the definitions (4.3) and (4.4). Moreover, integrating by parts (4.4) we get the inequalities

$$2\Psi_M(x) \leqslant U_M(x) \leqslant G(\infty) + 2\Psi_M(x), \quad (x \geqslant 0).$$

Thus $U_M \sim \Psi_M$, which completes the proof.

LEMMA 4.2. *Let $\{f_n\}$ be a sequence of Borel simple functions on I. The sequence $\{f_n\}$ converges to 0 in $\mathcal{L}(M)$ if and only if it converges to 0 in $L(\Psi_M)$.*

Proof. Suppose that $\{f_n\}$ converges to 0 in $\mathcal{L}(M)$. Then for every positive number C, $\{C^{-1} \int_I f_n(s)\, M(ds)\}$ tends to 0 in probability. Hence and from (4.2) it follows that $\lim_{n\to\infty} \int_I T_M(tC^{-1} f_n(s))\, ds = 0$ uniformly in every finite interval. Thus $\lim_{n\to\infty} \int_I \int_0^1 T_M(tC^{-1} f_n(s))\, dt\, ds = 0$ and, consequently, by Lemma 4.1 $\lim_{n\to\infty} \int_I \Psi_M(C^{-1} f_n(s))\, ds = 0$. Since $\|f\|_{\Psi_M} \leqslant C$ if $R_{\Psi_M}(C^{-1} f) < C$, the last equation implies the relation $\overline{\lim}_{n\to\infty} \|f_n\|_{\Psi_M} \leqslant C$. Thus $\lim_{n\to\infty} \|f_n\|_{\Psi_M} = 0$ because of the arbitrariness of C. Now suppose that $\{f_n\}$ converges to 0 in $L(\Psi_M)$. Since, by Lemma 2.1, $\Psi_M \sim U_M$, we may assume without loss of generality that $\|f_n\|_{U_M} < 1$ $(n = 1, 2, \ldots)$. Then the inequality $R_{U_M}(f_n) \leqslant \|f_n\|_{U_M}$ $(n = 1, 2, \ldots)$ holds. By Lemma 4.1 for every $a > 0$ we have the inequality

$$\max_{0 \leqslant t \leqslant a} \int_I T_M(tf_n(s))\, ds \leqslant c_1(a)\, R_{U_M}(f_n), \quad (n = 1, 2, \ldots).$$

Hence and from 4.2 follows that the sequence $\{\Psi_{f_n}\}$ of characteristic function converges to 1 uniformly in every finite interval. Thus $\{\int_I f_n(s)\, M(ds)\}$ converges to 0 in probability and, consequently, $\{f_n\}$ converges to 0 in $\mathcal{L}(M)$. The Lemma is thus proved.

Now we shall prove the main theorem.

THEOREM 4.1. *Let M be a symmetric homogeneous random measure. Then there exists a function $\Phi \in K_0$ such that $\mathcal{L}(M) = L(\Phi)$. Moreover $\Phi \sim \Psi_M$, where $\Psi_M(x) = \int_{1/x}^{\infty} \frac{G(u)}{u^3} du$ and G is the Lévy—Khinchine function corresponding to M. Conversely, for every function $\Phi \in K_0$ there exists a symmetric homogeneous random measure M such that $\mathcal{L}(M) = L(\Phi)$.*

Proof. Let M be a symmetric homogeneous random measure. Since $\Psi_M(\sqrt{x}) = \frac{1}{2}\int_0^x G\left(\frac{1}{\sqrt{u}}\right) du$, the function $\Psi_M(\sqrt{x})$ is concave and, consequently, $\Psi_M \in K_0$. Hence, in particular, it follows that Ψ_M satisfies the Δ_2-condition. Thus the set of all Borel simple functions is dense in both complete spaces $\mathcal{L}(M)$ and $L(\Psi_M)$. Now the equation $\mathcal{L}(M) = L(\Psi_M)$ is a consequence of Lemma 4.2.

Suppose that $\Phi \in K_0$. Of course we may assume without loss of generality that the function $\Phi(\sqrt{x})$ is concave. Then it can be written in the form $\Phi(\sqrt{x}) = \int_0^x q(u)\, du$, where the function q is monotone non-increasing, continuous on the right and non-negative. Put $G(0) = 0$ and $G(x) = \min(1, q(x^{-2}))$ $(x > 0)$. The function G is monotone non-decreasing, bounded, and continuous on the left. Moreover $G(\infty) > 0$ because the function Φ vanishes at the origin only. We already know that the function G is the Lévy—Khinchine function for a symmetric homogeneous random measure M. By a simple computation we obtain the relation $\Phi \sim \Psi_M$ where Ψ_M is defined by (4.5). Thus $\mathcal{L}(M) = L(\Phi)$ which completes the proof of the Theorem.

For some random measures M the space $\mathcal{L}(M)$ is even a Banach space. For instance, if M is a stable symmetric measure with the characteristic function $\varphi_{M(E)}(t) = \exp(-|E|\,|t|^p)$ $(1 \leqslant p \leqslant 2)$, then $\mathcal{L}(M)$ is an L^p-space. It is known that an Orlicz space $L(\Phi)$ is a Banach space if and only if the function Φ is equivalent to a convex function from K. Moreover, Φ is equivalent to a convex function from K if and only if for some constants $a, b > 0$ and $x_0 \geqslant 0$ the inequality $\frac{\Phi(y)}{y} \geqslant a\frac{\Phi(bx)}{x}$ $(y \geqslant x \geqslant x_0)$ holds ([6], [7]). Hence and from Theorem 4.1 we get the following

COROLLARY. *Let M be a symmetric homogeneous random measure with the Lévy—Khinchine function G. The space $\mathcal{L}(M)$ is a Banach space if and only if for some positive constants c and y_0 the inequality*

$$x\int_x^{\infty} \frac{G(u)}{u^3}\, du \geqslant cy\int_y^{\infty} \frac{G(u)}{u^3}\, du \quad (0 < x \leqslant y \leqslant y_0) \textit{ holds.}$$

INSTITUTE OF MATHEMATICS, POLISH ACADEMY OF SCIENCES
(INSTYTUT MATEMATYCZNY, PAN)

REFERENCES

[1] J. L. Doob, Trans. Am. Math. Soc. **42** (1937), 107—140.
[2] — , *Stochastic processes*, New York, 1953.
[3] N. Dunford and J. T. Schwartz, *Linear operators. I*, New York, 1958.
[4] P. R. Halmos, *Measure theory*, New York, 1950.
[5] M. Loève, *Probability theory*, Toronto, 1955.
[6] W. Matuszewska, *On general Orlicz spaces*, Bull. Acad. Polon. Sci., Sér. sci. math., astronom. et phys., **8** (1960), 349—353.
[7] W. Matuszewska and W. Orlicz, Studia Math., **21** (1961), 107—115.
[8] J. Musielak and W. Orlicz, ibid., **19** (1959), 49—65.
[9] A. Prékopa, Acta Math. Acad. Scient. Hung., **7** (1956), 215—262.
[10] — , ibid., **8** (1957), 337—374.
[11] — , ibid., **8** (1957), 375—400.