Central Physics Observatory
Petrograd, Vasilyevsky island, 23 line, house 2.
Professor A. Friedmann

Very learned Professor, `[directed to Weyl]`

During the careful study of your inspiring book "Space Time Matter", I hit on two difficulties in the geometry that you have developed; my efforts at overcoming these led me to some results, which I would like to communicate to you.

In your definition of the affine vector connection of space the quantities Γ^i_{rs} were introduced, which give the parallel transport (along a curve) of the (contravariant) vectors ξ^i through the equations

$$\frac{d\xi^i}{dt} = -\Gamma^i_{rs}\xi^r\frac{dx_s}{dt}.$$

For covariant vectors they determine the parallel transport through the equations

$$\frac{d\eta_i}{dt} = -\Gamma^\alpha_{is}\eta_\alpha\frac{dx_s}{dt}.$$

At first there seems to be no reason not to determine the parallel transport of the covariant vectors through the general equation $\frac{d\xi^i}{dt} = -G^i_{rs}\xi^r\frac{dx_s}{dt}$; then there must be a relation between Γ^α_{is} and G^α_{is}. Because the contravariant vectors are closely tied to the direction of the curve, it is entirely natural that the covariant vectors are closely tied to the normal to the hypersurface.

Analogously to your flat (geodesic) lines one can establish the notion of a "flat" hypersurface, if one calls a hypersurface "flat" when the normal upon translation along the surface remains parallel to itself; upon closer inspection of the possibility of such a hypersurface it follows, that if G^α_{is} is symmetric in the lower indices the curvature tensor of G^α_{is} is identically zero; hence such hypersurfaces exist only exceptionally.

More interesting results follow from the consideration of conically analogous hypersurfaces constructed from the straight lines through a point (geodesic lines).

If one makes the requirement that translation of the normals to this cone is translation along the generating parallels, one obtains the equations:

$$G^i_{rs}, \quad r \text{ not } = i$$
$$G^i_{is} = \Gamma^i_{is} + \psi_{s'} \quad \psi_s \text{ independent of the index } i,$$

from which it follows that ψ_s is a covariant vector.

In this way a vector connection becomes possible that is more general than the one you considered and which in addition to Γ^i_{rs} contains also the covariant vector ψ_s. It is clear, as I remark elsewhere, that the consideration of this case can modify your so significant developments of Riemann's thoughts about the connection between space and physics.

The second difficulty was for me the question, why in the definition of the metric connection you set dl (in your notation) proportional to l. I have clarified this question for myself in the following way. The angle ω between the contravariant vectors ξ^i, η^i and the angle Ω between the covariant vectors ξ_i, η_i is defined through the formulas

$$\cos\omega = \frac{g_{ik}\xi^i\eta^k}{\sqrt{g_{\alpha\beta}\xi^\alpha\xi^\beta}\sqrt{g_{\alpha\beta}\eta^\alpha\eta^\beta}}, \quad \cos\Omega = \frac{g^{ik}\xi_i\eta_k}{\sqrt{g^{\alpha\beta}\xi_\alpha\xi_\beta}\sqrt{g^{\alpha\beta}\eta_\alpha\eta_\beta}};$$

the reason why the angle between the covariant vectors is defined through the second of these formulas becomes understandable from the consideration of the angles between two hypersurfaces (see for example Bianchi, Lectures, page 332).

Let us require that the angle between the vectors is conserved upon parallel transport. A direct calculation gives

$$\frac{d}{dt}\left(g_{ik}\xi^i\eta^k\right) = \frac{dx_s}{dt}\xi^i\eta^k A_{iks},$$

and our requirement ($\frac{d\omega}{dt} = 0$) leads after some intermediary calculations to the equations

$$g_{lm}A_{iks} = g_{ik}A_{lms},$$

that is to say, $A_{iks} = -g_{ik}\phi_s$, where ϕ_s is independent of the indices i, k. So we obtain

$$\frac{d}{dt}\left(g_{ik}\xi^i\eta^k\right) = \left(-\phi_s\frac{dx_s}{dt}\right)\left(g_{ik}\xi^i\eta^k\right),$$

and that is the metric connection introduced by you. From the consideration of the equation $\frac{d\Omega}{dt} = 0$ (upon parallel transport) one arrives at:

$$\frac{d}{dt}\left(g^{ik}\xi^i_i\eta^k_k\right) = \left(-\phi'_s\frac{dx_s}{dt}\right)\left(g^{ik}\xi_i\eta_k\right)$$

So the generalization of your principle of the vectorial and metric connection leads in addition to the fundamental form also to the three vectors ϕ_s, ϕ'_s and ψ_s, while you had only one. Further calculations (starting from the expression for $\frac{\partial g^{ik}}{\partial x_s}$) lead to the following relation between these three vectors: $\phi_s + \phi'_s + 2\psi_s = 0$, so only two vectors remain independent, for example, ϕ_s and ψ_s. The first determines the metric connection of space, the second gives a relation between parallel transport of co- and contravariant vectors.

Because space is determined by physical quantities, one might perhaps assume that also the new covariant vector (or the new linear form $\psi_s dx_s$) has a physical significance; then it is permissible to think that the relation between space and matter can not only be considered in the sense of Mie's theory, but also independently from it, if one introduces not 4 but 8 functions — two covariant vectors.

I beg your pardon for this somewhat long letter, but I think that this generalization is not without interest.

As soon as I can, I will send you a detailed exposition of my considerations; and I would ask you, if this is possible for you, to have it published somewhere. I would also be very pleased, if you would show me the kindness to let me know your opinion.

Together with J. D. Tamarkin I have worked on the propagation of gravitational waves. We have applied the general method of Hadamard and obtained the interesting result, that the gravity wave (discontinuity of the second kind) can not move faster than the speed of light.

A. Friedmann