THEORY OF THE RADIATION FROM ORIENTED NUCLEI

J. A. M. COX

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Proefschrift

TER VERKRIJGING VAN DE GRAAD VAN DOCTOR IN DE WIS- EN NATUURKUNDE AAN DE RIJKS-UNIVERSITEIT TE LEIDEN, OP GEZAG VAN DE RECTOR MAGNIFICUS MR J. M. VAN BEMMELEN, HOOGLERAAR IN DE FACULTEIT DER RECHTSGELEERDHEID, PUBLIEK TE VERDEDIGEN OP WOENSDAG 24 MAART 1954 TE 16 UUR

DOOR

JEAN AUGUST MARIA COX

GEBOREN TE HEERLEN

Promotor: PROF. DR S. R. DE GROOT

Aan mijn moeder
Aan de nagedachtenis van mijn vader
Aan mijn vrouw

Op verzoek van de Faculteit volgen hier enkele gegevens over mijn studie.

In 1942 legde ik het eindexamen H.B.S.-B. af aan de Rijks H.B.S. te 's-Hertogenbosch. In de periode 1942–1945 was ik als assistent werkzaam op het radiobuizenontwikkelingslaboratorium der N.V. Philips te Eindhoven. In 1945 begon ik mijn studie aan de Universiteit te Utrecht en legde in Mei 1946 het candidaatsexamen wis- en natuurkunde (d) af. In Maart 1948 volgde het doctoraal examen met hoofdvak theoretische natuurkunde en bijvakken wiskunde en mechanica.

Gedurende enige tijd hierna werkte ik onder Prof. S. R. de Groot aan een research-program voor bêta-radioactiviteit.

Van 1948–1952 was ik als leraar werkzaam aan het Stedelijk Gymnasium te Maastricht. Tegelijkertijd verrichtte ik met Prof. S. R. de Groot en Dr H. A. Tolhoek researchwerk op kernphysisch gebied, o.m. over het vloeistofdruppelmodel van de atoomkern en over de gamma-straling uitgezonden door gerichte kernen. Laatstgenoemd onderzoek werd begonnen op verzoek van Prof. C. J. Gorter in verband met experimenten op het Kamerlingh Onnes-laboratorium te Leiden.

In 1952 volgde mijn benoeming aan het Van der Waals-laboratorium te Amsterdam bij Prof. A. Michels, waar ik thans werkzaam ben op het gebied der molecuulphysica.

In de laatste jaren verschenen een aantal publicaties over kernstraling uitgezonden door gerichte atoomkernen (Physica **18** (1952) 357, 359, 1262; **19** (1953) 101, 673, 683, 1178, 1119, 1123).

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CONTENTS

INTRODUCTION

If we consider an ensemble of radioactive nuclei of which the spin directions have not a random distribution in space, we may expect that the emitted radiations will show angular effects. First of all, the angular distribution of the emitted radiation will deviate from spherical symmetry. Besides this effect, (partial) polarization of the radiation will occur. If we have two radiations emitted in cascade, the directional correlation will be different from the correlation in the case of randomly oriented nuclei.

In this thesis a theoretical treatment is given of the above mentioned geometrical properties of the radiations (especially γ-radiation) from oriented nuclei. The theoretical aspects of the methods by which the nuclei are oriented will not be considered here.

In chapter I we shall calculate the angular distribution and polarization of the emitted radiations. In this chapter we shall assume that the nuclear orientation has rotational symmetry. To obtain formulae which demonstrate the relevant physical features in a simple way, it is of great importance to choose suitable parameters for the description of the nuclear orientation and the polarization of the radiation. Two separate sections are dedicated to this subject. For γ-dipole and quadrupole radiation, formulae will be given which can be evaluated by simple substitutions.

In chapter II we shall study the influence of a β-transition, preceding the emission of the observed γ-radiation, on the angular distribution and polarization of the γ-radiation. This effect is of importance as it often occurs in practice .As an example, the calculation of the angular distribution of the γ-radiation from oriented ^{60}Co nuclei will be outlined, taking into account the effect of the preceding β-transition. In this chapter we shall also summarize which quantities of physical interest can in principle be found by measuring the radiation from oriented nuclei.

In chapter III we shall drop the assumption of rotational symmetry made in chapter I. Then the angular distribution will be calculated for the radiation emitted by an arbitrary ensemble of radioactive nuclei.

Using the results of chapter III we shall give in chapter IV the directional correlation of successive radiations emitted by an ensemble of oriented nuclei assuming again an axis of rotational symmetry for the initial ensemble of nuclei. For γ-dipole and quadrupole radiation explicit formulae will be given suitable for numerical evaluation.

Chapter I

ANGULAR DISTRIBUTION AND POLARIZATION OF GAMMA RADIATION EMITTED BY ORIENTED NUCLEI

Synopsis

The angular distribution and polarization of γ-radiation emitted by oriented nuclei is calculated assuming pure multipole transitions. For the description of the orientation of the nuclei $2j$ independent parameters f_k are used; these are essentially the statistical tensors introduced by F a n o (j is the spin quantum number of the nuclei). The state of polarization of the γ-radiation is characterized by the degree of polarization P and a real three-dimensional polarizationvector $\boldsymbol{\xi}_0$. If these quantities f_k, P and $\boldsymbol{\xi}_0$ are used, the formulae take a relatively simple form. For the cases of dipole and quadrupole radiation expressions are given which are suitable for numerical calculation by easy substitutions.

§ 1. *Introduction.* If we have an ensemble of oriented nuclei, the angular distribution of γ-radiation emitted by these nuclei no longer has spherical symmetry. S p i e r s [1]) discussed this effect and S t e e n b e r g [2]) extended his considerations, both giving formulae which are practical only, in the case of small nuclear orientation. In the following sections we shall derive explicit formulae for the angular distribution of γ-radiation from oriented nuclei, valid for any degree of orientation of the nuclei. Furthermore we shall calculate the polarization of the emitted γ-radiation.

The theory of γ-radiation from *oriented nuclei* is closely related to the theory of angular correlation of pairs of successive γ-quanta from nuclei *oriented at random*. In the latter case the angular correlation function can be considered to be the angular distribution function for the second radiation emitted by nuclei which have been oriented by the emission of the first radiation in a fixed direction. A short review of the relation between these problems will be given in this section. The theory of angular correlation of successive γ-quanta emitted by nuclei oriented at random was given by Hamilton[3]). Recent contributions have been made by Falkoff, Ling, Uhlenbeck[4])[5]), Racah[6]), Lloyd[7]) and Alder[8]).

We shall begin with some general formulae concerning the angular distribution of radiation from oriented nuclei. We assume the nuclei to be oriented in such a way that an axis $\boldsymbol{\eta}$ of rotational symmetry exists. The nuclear angular momentum quantum number and magnetic quantum number are j and m respectively (m determines the component of the nuclear angular momentum in the direction $\boldsymbol{\eta}$, which we call the axis of quantization). The orientation of the nuclei is then characterized by the numbers a_m, these being the probabilities of the states specified by j, m ($\Sigma a_m = 1$).

$I_{j_i}^{m_i}(\vartheta)$ is the angular distribution of the radiation from a nucleus in the state j_i, m_i.

$W(\vartheta)$ is the angular distribution of the observed radiation.

ϑ is the angle between the direction of emission of the observed radiation and $\boldsymbol{\eta}$.

Now $W(\vartheta)$ is given by

$$W(\vartheta) = \Sigma_{m_i} a_{m_i} I_{j_i}^{m_i}(\vartheta). \tag{1}$$

If the angular distribution of radiation with angular momentum quantum number L and magnetic quantum number M is denoted by $F_L^M(\vartheta)$, we can express $I_{j_i}^{m_i}(\vartheta)$ in terms of $F_L^M(\vartheta)$ by

$$I_{ji}^{m_i}(\vartheta) = \Sigma_M \, G_{m_i,m_i-M}^{j_iLj_f} \, F_L^M(\vartheta). \tag{2}$$

The $G_{m_i,m_i-M}^{j_iLj_f}$ are the squared transformation coefficients for the addition of angular momenta (see formula 60).

In general there will be a β or γ-transition $(j_0, m_0) \rightarrow (j_i, m_i)$, which precedes the radiation under consideration. In this case the a_{m_0} give the initial orientation of the nuclei and we have to calculate the relative populations a_{m_i} of the levels m_i in order to compute $W(\vartheta)$ for the radiation emitted by the nuclei with spin j_i. If $P(m_0, m_i)$ gives the relative transition probability for the transition $(j_0, m_0) \rightarrow (j_i, m_i)$, the expression for a_{m_i} becomes

$$a_{m_i} = \Sigma_{m_0} a_{m_0} P(m_0, m_i), \tag{3}$$

$$\Sigma_{m_i} P(m_0, m_i) = 1. \tag{4}$$

We add some formulae on the angular correlation of successive radiations emitted by nuclei which are oriented at random.

$P(m_1, m_2, \vartheta)$ is the angular distribution of the radiation emitted in the transition $(j_1, m_1) \rightarrow (j_2, m_2)$. ϑ is the angle between the direction of the radiation and the axis of quantization. This quantity $P(m_1, m_2, \vartheta)$ is related to $P(m_1, m_2)$, $I_{j_1}^{m_1}(\vartheta)$ and $F_L^M(\vartheta)$ by

$$P(m_1, m_2) = \int P(m_1, m_2, \vartheta)\, \mathrm{d}\Omega, \tag{5}$$

$$I_{j_1}^{m_1}(\vartheta) = \Sigma_{m_2} P(m_1, m_2, \vartheta), \tag{6}$$

$$P(m_1, m_2, \vartheta) = G_{m_1,m_2}^{j_1 L j_2} F_L^M(\vartheta),\ (m_1 = m_2 + M). \tag{7}$$

The angular correlation function $W(\vartheta)$, giving the relative probability for the angle ϑ between the successive radiations, is given by (cf. [5])

$$W(\vartheta) = C\, \Sigma_{m_0,m_i,m_f} P(m_0, m_i, \vartheta = 0) . P(m_i, m_f, \vartheta). \tag{8}$$

Here the successive states of the nucleus are (j_0, m_0), (j_i, m_i) and (j_f, m_f). With (6) we can rewrite (8) as

$$W(\vartheta) = \Sigma_{m_i} a_{m_i} I_{j_i}^{m_i}(\vartheta), \tag{9}$$

$$a_{m_i} = C\, \Sigma_{m_0} P(m_0, m_i, \vartheta = 0). \tag{10}$$

These formulae allow the following interpretation. The direction $\vartheta = 0$ of the first radiation is an axis $\boldsymbol{\eta}$ of rotational symmetry in the problem of the angular distribution of the second radiation. The first radiation causes an orientation of the nuclei (j_i, m_i) given by (10). From this point of view the angular correlation is a special case of the angular distribution of radiation from oriented nuclei. For a derivation of these formulae we refer to [3] and [5]. It is essential in our interpretation of the formulae that (8) allows a "splitting of the process into two parts". Cf. [9] for a discussion of this point.

In § 2 is discussed how the orientation of atomic nuclei can be characterized. In § 3 the use of different parameters to characterize the polarization of γ-radiation is treated. With §§ 1, 2 and 3 as a basis the calculation of the angular distribution and polarization of γ-radiation of oriented nuclei is rather straightforward. This is discussed in § 4. With the aid of more advanced mathematics, namely, a method developed by R a c a h [10], we can derive formulae for arbitrary multipole order in § 5. Though the results of § 5 include the results of § 4, we have thought it worth while to discuss this first method, as it gives a clearer understanding of some features and relations in this field. Explicit formulae for the angular distribution are given in § 6, for the polarization in § 7 (in these expressions the general expressions have been evaluated as far as possible). A short note containing some of the results has appeared earlier [11]). Later we shall discuss the ways in which these formulae can be used for the treatment of experiments with oriented nuclei [12]).

§ 2. *The description of the orientation of atomic nuclei by the parameters f_k.* Generally the state of orientation of an ensemble of nuclei, all with angular momentum j (or of one nucleus of which our knowledge is incomplete), cannot be described by means of a single wave function. We therefore use a density matrix (or statistical operator) ϱ with $(2j + 1)^2$ matrix elements (compare e.g. [13]), part II and IV). ϱ is hermitian and normalized to

$$\Sigma_m \varrho_{mm} = 1. \tag{11}$$

With each state $$\psi = \Sigma_m c_m \psi_m, \tag{12}$$

we associate a matrix $$(\varrho_\psi)_{mm'} = c_m c^*_{m'}. \tag{13}$$

The probability of finding a system described by the density matrix ϱ in the state ψ is then given by

$$W = \mathrm{Tr}\,(\varrho\varrho_\psi). \text{ (Tr: Trace)} \tag{14}$$

If an axis $\boldsymbol{\eta}$ of rotational symmetry exists, and $\boldsymbol{\eta}$ has been chosen as axis of quantization, then ϱ is a diagonal matrix (compare [9])). Now the probability of finding a state ψ_m is given by

$$a_m = \mathrm{Tr}(\varrho\varrho_{\psi_m}) = \varrho_{mm}. \tag{15}$$

In this case ϱ is determined by $2j + 1$ numbers a_m. As $\Sigma_m a_m = 1$ on account of (11), there are only $2j$ independent numbers a_m which characterize the orientation. As in the results of the calculations in the following sections a_m will appear always in combinations of the form $\Sigma_m m^k a_m$ (often called moments), we therefore define $2j$ independent combinations of this form, which are equivalent to the set of $2j$ numbers a_m.

$$f_k = \Sigma_{\nu=0}^{k} a_{k,\nu}\, \Sigma_m m^\nu a_m, \tag{16a}$$

$$f_k = 0 \text{ if } a_m = \Sigma_{p=0}^{k-1} A_p m^p, \tag{16b}$$

$$a_{k,k} = j^{-k}. \tag{16c}$$

By taking $a_m = m^p$ ($p = 0 \ldots . k - 1$) in (16b) we get k linearly independent relations for the $k+1$ coefficients $\alpha_{k,0} \ldots . \alpha_{k,k}$. It is easily seen that these coefficients become entirely determined by the additional condition (16c). This condition has been chosen in such a way that the f_k for totally oriented nuclei remain finite if we let $j \to \infty$ (cf. (19a).... (19d). Hence the f_k are uniquely defined by (16). If an arbitrary orientation is given by a_m, the dependence of a_m on m can always be expressed by a polynomial of degree $2j$:

$$a_m = \Sigma_{p=0}^{2j} A_p \, m^p. \tag{17}$$

Hence we obtain the following interesting property of the f_k directly from the definition (16): *all f_k with $k \geqslant 2j+1$ vanish identically.*

Since $f_0 = 1$ cannot vary there are $2j$ parameters f_k left which are independent and suffice for a complete description of the orientation. These f_k will be used as they give simple forms to our formulae. Explicit expressions for f_1, f_2, f_3 and f_4 are:

$$f_1 = j^{-1} \, \Sigma_m \, m a_m, \tag{18a}$$

$$f_2 = j^{-2} \, [\Sigma_m \, m^2 a_m - \tfrac{1}{3} j(j+1)], \tag{18b}$$

$$f_3 = j^{-3} \, [\Sigma_m \, m^3 \, a_m - \tfrac{1}{5}(3j^2 + 3j - 1) \, \Sigma_m \, m a_m], \tag{18c}$$

$$f_4 = j^{-4} [\Sigma_m m^4 a_m - \tfrac{1}{7}(6j^2 + 6j - 5) \, \Sigma_m m^2 a_m + \tfrac{3}{35} j(j-1)(j+1)(j+2)]. \tag{18d}$$

If the nuclei are totally oriented ($a_m = \delta_{mj}$), we calculate from (18)

$$f_1 = 1 \tag{19a}$$

$$f_2 = (2j-1)/3j \tag{19b}$$

$$f_3 = (j-1)(2j-1)/5j^2 \tag{19c}$$

$$f_4 = 2(j-1)(2j-1)(2j-3)/35j^3. \tag{19d}$$

For numerical calculations it can be useful to approximate a distribution by a polynomial in order to calculate the f_k. If we have for example a Boltzmann distribution

$$a_m = C' \exp(\beta m) \quad (\text{with } \beta = \mu B/kTj), \tag{20a}$$

we can write approximately

$$a_m = C\,(1 + \beta m + \tfrac{1}{2}\beta^2 m^2 + \tfrac{1}{6}\beta^3 m^3 + \tfrac{1}{24}\beta^4 m^4) \text{ if } \beta j \ll 1. \tag{20b}$$

Now if we have in general a distribution

$$a_m = C\,(1 + A_1 m + A_2 m^2 + A_3 m^3 + A_4 m^4), \tag{21}$$

it is readily shown that

$$f_1 = [\tfrac{1}{3}(j+1) \, A_1 + \tfrac{1}{15}(j+1)(3j^2 + 3j - 1) \, A_3] \, n, \tag{22a}$$

$$f_2 = [\tfrac{1}{45j}(2j-1)(j+1)(2j+3)A_2 + \tfrac{1}{315j}(2j-1)(j+1)(2j+3)$$
$$(6j^2+6j-5)A_4]n, \quad (22b)$$
$$f_3 = (\tfrac{1}{175}j^{-2})(j-1)(2j-1)(j+1)(2j+3)(j+2)A_3\,n, \quad (22c)$$
$$f_4 = \tfrac{4}{(105)^2}j^{-3}(2j-3)(j-1)(2j-1)(j+1)(2j+3).$$
$$.(j+2)(2j+5)A_4\,n. \quad (22d)$$
$$\{(2j+1)\,C\}^{-1} = 1/n = 1+\tfrac{1}{3}j(j+1)\,A_2+\tfrac{1}{15}j(j+1)(3j^2+3j-1)A_4 \quad (22e)$$

We can derive an explicit expression for the f_k by comparing them with the statistical tensors introduced by F a n o [14]) according to the definition

$$\langle|(jj)kq\rangle = \Sigma_{mm'}\langle m|\varrho|m'\rangle(-1)^{j-m}\langle jm', j-m|(jj)kq\rangle, \quad (23)$$

where $\langle m|\varrho|m'\rangle$ are the matrix elements of the density matrix ϱ, and $\langle jm', j-m|(jj)kq\rangle$ are the transformation coefficients for the addition of angular momenta. In the case in which the ensemble of nuclei has an axis of rotational symmetry $\boldsymbol{\eta}$ we write

$$\bar{f}_k = \langle\,|(jj)\,k0\rangle = \Sigma_m\langle m|\varrho|m\rangle(-1)^{j-m}\langle jm, j-m|(jj)\,k0\rangle. \quad (24)$$

The density matrix ϱ is then entirely determined by the probabilities $a_m = \langle m|\varrho|m\rangle$ and we can write

$$\bar{f}_k = \Sigma_m(-1)^{j-m}\langle jm, j-m|(jj)\,k0\rangle\,a_m \quad (25)$$

We now show that the $\bar{f}_k$ introduced in this way differ only by a constant factor from the f_k defined by (16), i.e.,

$$f_k = w_k(j)\bar{f}_k. \quad (26)$$

To prove (26), we observe the $\bar{f}_k$ to be of the form

$$\bar{f}_k = \Sigma_m R_k(m)\,a_m, \quad (27a)$$

$$R_k(m) = (-1)^{j-m}\langle jm, j-m|(jj)\,k0\rangle. \quad (27b)$$

$R_k(m)$ is a polynomial in m of degree k, as may be seen from the explicit formula for $\langle jmj-m|(jj)\,k0\rangle$ (Cf. [10]) formula (16)),

$$\langle jmj-m|(jj)\,k0\rangle = \left[\frac{(2k+1)\,(2j-k)!}{(2j+k+1)!}\right]^{\frac{1}{2}}(k!)^2\,(j+m)!\,(j-m)!\times$$
$$\times\,\Sigma_z(-1)^z[z!\,(2j-k-z)!\,\{(j-m-z)!\}^2\{(k-j+m+z)!\}^2]^{-1}, \quad (27c)$$

by rewriting with this expression formula (27b) as

$$R_k(m) = \left[\frac{(2k+1)\,(2j-k)!}{(2j+k+1)!}\right]^{\frac{1}{2}}\binom{2k}{k}[m^k + u_1m^{k-1} + \ldots u_0]. \quad (28)$$

From the orthogonality relations for $\langle jmj—m|(jj)\, k0\rangle$ it follows that

$$\Sigma_m R_k(m)\, R_p(m) = 0, \qquad p \leqslant k-1. \tag{29}$$

From (28) and (29) it is easily deduced that

$$\Sigma_m R_k(m) m^p = 0, \qquad p \leqslant k-1. \tag{30}$$

If a_m is given by $a_m = \Sigma_{p=0}^{k-1} A_p m^p$ it follows from (30) that

$$\bar{f}_k = \Sigma_m R_k(m)\, a_m = 0. \tag{31}$$

So the $\bar{f}_k$ are of the form (16*a*) and satisfy the condition (16*b*). Hence the $\bar{f}_k$ and f_k can only differ by a constant factor, which may be obtained from (28) and (16*c*)

$$f_k/\bar{f}_k = w_k(j) = \binom{2k}{k}^{-1} j^{-k} \left[\frac{(2j+k+1)!}{(2k+1)\,(2j-k)!}\right]^{\frac{1}{2}}. \tag{32}$$

From (26), (27*a*), (27*b*) and (27*c*) an explicit expression for f_k follows

$$f_k = \binom{2k}{k}^{-1} j^{-k}\, \Sigma_m \Sigma_{\nu=0}^{k} (-1)^\nu \frac{(j-m)!\,(j+m)!}{(j-m-\nu)!\,(j+m-k+\nu)!} \binom{k}{\nu}^2 a_m. \tag{33}$$

For totally oriented nuclei ($a_m = \delta_{mj}$) f_k has the value (a generalization of (19*a*) .... (19*d*))

$$f_k = \binom{2k}{k}^{-1} j^{-k} \frac{(2j)!}{(2j-k)!} \tag{34}$$

§ 3. *Characterization of the polarization of electro-magnetic radiation.* For a plane electro-magnetic wave which is propagated in the direction $\mathbf{k}(|\mathbf{k}| = 2\pi/\lambda)$ we can write for the complex vector potential $\mathbf{A}$ and electric field strength $\mathbf{E}$

$$\mathbf{A} = (c_1\mathbf{e}_1 + c_2\mathbf{e}_2)\, A, \qquad A = \hat{A} \exp.\, i\,(\mathbf{k}.\mathbf{r} - \omega t), \tag{35}$$

$$\mathbf{E} = (c_1\mathbf{e}_1 + c_2\mathbf{e}_2)\, E, \qquad E = -(\partial A/\partial t)/c = i|\mathbf{k}|A. \tag{36}$$

Here $\mathbf{k}/|\mathbf{k}|$, $\mathbf{e}_1$ and $\mathbf{e}_2$ are mutually perpendicular unit vectors. We assume c_1 and c_2 to be normalized to

$$|c_1|^2 + |c_2|^2 = 1. \tag{37}$$

The state of polarization is characterized by the complex vector

$$\mathbf{c} = (c_1, c_2). \tag{38}$$

Essentially two (real) parameters (e.g. the ratio $|c_1|a/|c_2|$ and the phase difference of c_1 and c_2) are needed for the description of a totally polarized wave (or photon). The state of polarization being described by $\mathbf{c}$, we define a density matrix $\varrho(\mathbf{c})$ ([13]) II formula 4):

$$\varrho(\mathbf{c}) = \begin{Vmatrix} |c_1|^2 & c_1 c_2^* \\ c_1^* c_2 & |c_2|^2 \end{Vmatrix}. \tag{39}$$

This special form of the density matrix occurs only in the case of total polarization. The states of polarization of a partially polarized beam of photons is described by a hermitian density matrix ϱ:

$$\varrho = \begin{Vmatrix} \varrho_{11} & \varrho_{12} \\ \varrho_{21} & \varrho_{22} \end{Vmatrix}, \tag{40a}$$

$$\varrho_{11} + \varrho_{22} = 1. \tag{40b}$$

In this case essentially three parameters are needed. Now the probability W of finding a photon in the state of polarization described by $\mathbf{c}$ is given by

$$W = \mathbf{c}^* \, \varrho \, \mathbf{c} = \mathrm{Tr}\,[\varrho\varrho(\mathbf{c})]. \tag{41}$$

If $\varrho = \varrho(\mathbf{c}')$ (totally polarized beam), (41) becomes

$$W = \mathrm{Tr}\,[\varrho(\mathbf{c}')\,\varrho(\mathbf{c})] = |c_1' c_1^* + c_2' c_2^*|^2. \tag{42}$$

Another description of the state of total polarization, discussed by Fano [15]), makes use of parameters α and β. These are related to $\mathbf{c}$ by

$$c_2/c_1 = (\sin\alpha\cos\beta + i\cos\alpha\sin\beta)/(\cos\alpha\cos\beta - i\sin\alpha\sin\beta) =$$
$$= (\sin 2\alpha \cos 2\beta + i \sin 2\beta)/(1 + \cos 2\alpha \cos 2\beta). \tag{43}$$

α and β are the angles which determine the setting of an ideal analyzer (consisting of a $\lambda/4$ plate and a Nicol prism) with respect to $\mathbf{e}_1$ for the case of maximum transmission. A real three dimensional vector $\boldsymbol{\xi}$, the polarization vector, can be defined as follows with the aid of α and β:

$$\begin{aligned} \xi_1 &= \cos 2\beta \cos 2\alpha, \\ \xi_2 &= \cos 2\beta \sin 2\alpha, \\ \xi_3 &= \sin 2\beta. \end{aligned} \tag{44}$$

We can describe a state of total polarization by this vector $\boldsymbol{\xi}$. The relation between $\boldsymbol{\xi}$ and $\mathbf{c}$, following from (43) and (44), is

$$\begin{aligned} \xi_1 &= |c_1|^2 - |c_2|^2, \\ \xi_2 &= c_1 c_2^* + c_1^* c_2, \\ \xi_3 &= i\,(c_1 c_2^* - c_1^* c_2). \end{aligned} \tag{45}$$

In the case of a partially polarized beam, which is described by the density matrix ϱ (40), we can always write ϱ in the form

$$\varrho = \tfrac{1}{2}(1 - P)\begin{Vmatrix} 1 & 0 \\ 0 & 1 \end{Vmatrix} + P \begin{Vmatrix} |c_1|^2 & c_1 c_2^* \\ c_1^* c_2 & |c_2|^2 \end{Vmatrix}, \tag{46a}$$

$$0 \leqslant P \leqslant 1. \tag{46b}$$

This means a decomposition of ϱ into two parts, corresponding to a totally polarized and an unpolarized part. P is called the degree of polarization of the beam. $\boldsymbol{\xi}_0$ is the polarization vector of the beam and is determined by $\mathbf{c}$ of formula (46a). To describe the polarization of an arbitrary beam we shall use P and $\boldsymbol{\xi}_0$, containing again three real parameters. With (41) and (46) it follows that the probability W of finding a photon with polarization vector $\boldsymbol{\xi}$ in a beam described by P and $\boldsymbol{\xi}_0$ is given by

$$W = \tfrac{1}{2}(1 + P\,\boldsymbol{\xi}.\boldsymbol{\xi}_0). \tag{47}$$

For the special case of a totally polarized beam, with $P = 1$, we find a formula corresponding to (42):

$$W = \tfrac{1}{2}(1 + \boldsymbol{\xi}.\boldsymbol{\xi}_0). \tag{48}$$

The formalism which has been described, for the polarization of electromagnetic radiation, is to a high degree analogous to the formalism for the polarization of electrons, developed in [13]) part II and IV. Compare e.g., formula (47) with formula (70) of [13]) part IV.

§ 4. *A calculation method of the angular distribution and polarization of 2^L-pole γ-radiation (especially for $L = 1$ and $L = 2$).* The angular distribution of γ-radiation of a certain (L, M) pole character results from the spherical eigenwave solution of Maxwell's equation [16]) [17]) [4]). We have used the solutions for the vector potential (in the gauge of zero scalar potential) of the electric and magnetic (L, M) pole radiation in the form in which they are listed in [4]), table I (For the physical quantity $\mathbf{A}$ the real part of the complex quantity must be taken). The electric and magnetic field strengths are obtained from the vector potential $\mathbf{A}$ according to

$$\mathbf{E} = -(\partial\mathbf{A}/\partial t)/c, \tag{49}$$

$$\mathbf{B} = \operatorname{rot}\mathbf{A}. \tag{50}$$

The angular distribution of the radiation of a given (L, M) pole character is given by the magnitude of the Poynting vector $\mathbf{S}_L^M$.

If $\mathbf{n}$ is a unit vector in the direction of $\mathbf{S}_L^M$ then the magnitude is given by

$$\mathbf{n} \cdot \mathbf{S}_L^M = (ck^2/8\pi)\ \mathbf{A} \cdot \mathbf{A}^*. \tag{51}$$

We define

$$F_L^M(\vartheta) = 32\pi^3 r^2 c^{-1}\ \mathbf{n} \cdot \mathbf{S}_L^M. \tag{52}$$

Then $F_L^M(\vartheta)$ is normalized to

$$\int F_L^M(\vartheta)\ \mathrm{d}\Omega = 8\pi. \tag{53}$$

From (49), (51) and (52) it follows that

$$F_L^M(\vartheta) = 4\pi^2 r^2 k^2\ \mathbf{A} \cdot \mathbf{A}^*, \tag{54}$$

$$F_L^M(\vartheta) = 4\pi^2 r^2\ \mathbf{E} \cdot \mathbf{E}^*. \tag{55}$$

From the general expression for $\mathbf{A}$ one obtains an expression for $F_L^M(\vartheta)$ according to (54) (Y_L^M are the spherical harmonics):

$$F_L^M(\vartheta) = (4\pi/L(L+1))\ [2M^2|Y_L^M|^2 + (L-M)(L+M+1)|Y_L^{M+1}|^2 + \\ + (L+M)(L-M+1)\ |Y_L^{M-1}|^2]. \tag{56}$$

This expression becomes, for $L = 1$ and $L = 2$,

$$F_1^0(\vartheta) = 3\ (1 - \cos^2\vartheta), \tag{57a}$$

$$F_1^{\pm1}(\vartheta) = \tfrac{3}{2}\ (1 + \cos^2\vartheta), \tag{57b}$$

$$F_2^0(\vartheta) = \tfrac{5}{2}\ (6\cos^2\vartheta - 6\cos^4\vartheta), \tag{58a}$$

$$F_2^{\pm1}(\vartheta) = \tfrac{5}{2}\ (1 - 3\cos^2\vartheta + 4\cos^4\vartheta), \tag{58b}$$

$$F_2^{\pm2}(\vartheta) = \tfrac{5}{2}\ (1 - \cos^4\vartheta). \tag{58c}$$

These formulae apply for electric as well as magnetic multipole radiation. With these results we have calculated the angular distribution of 2^L-pole γ-radiation emitted by oriented nuclei according to the formula (compare (1) and (2)).

$$W(\vartheta) = \Sigma_{m_i,M}\ a_{m_i}\ G_{m_f m_i - M}^{j_i L j_f} \cdot F_L^M(\vartheta), \tag{59}$$

$$G_{m_f m_i - M}^{j_i L j_f} = |\langle j_f m_f\ LM |\ j_f L j_i m_i\rangle|^2. \tag{60}$$

In [18]), pages 76 and 77, one finds tables of transformation coefficients $\langle j_1 m_1 j_2 m_2 | j_1 j_2 jm\rangle$. Results obtained from (59), (60), and (57), and (58) for dipole and quadrupole radiation respectively, are given in § 6. Since

$$\Sigma_{m_2}\ |\langle j_1, m_1, j_2 m_2\ |\ j_1 j_2 jm\rangle|^2 = 1, \tag{61}$$

it follows from (53) and $\Sigma_m a_m = 1$, that $W(\vartheta)$ is normalized to

$$\int W(\vartheta)\ \mathrm{d}\Omega = 8\pi. \tag{62}$$

TABLE I

electric		magnetic		2^L-pole radiation; $kr \gg 1$. $C = \frac{1}{r}\left(\frac{2}{\pi}\right)^{\frac{1}{2}} (-i)^{L+1} e^{ik(r-ct)}$.
E_r	B_r	E_r	B_r	0
$-E_\varphi$	B_ϑ	E_ϑ	B_φ	$\left\{\frac{1}{2}\sqrt{\frac{(L-M)(L+M+1)}{L(L+1)}}\, Y_L^{M+1} \cos\vartheta\, e^{i\varphi} + \frac{M}{\sqrt{L(L+1)}}\, Y_L^M \sin\vartheta + \frac{1}{2}\sqrt{\frac{(L+M)(L-M+1)}{L(L+1)}}\, Y_L^{M-1} \cos\vartheta\, e^{i\varphi}\right\}.C$
E_ϑ	B_φ	E_φ	$-B_\vartheta$	$\left\{-\frac{i}{2}\sqrt{\frac{(L-M)(L+M+1)}{L(L+1)}}\, Y_L^{M+1} e^{-i\varphi} + \frac{i}{2}\sqrt{\frac{(L+M)(L-M+1)}{L(L+1)}}\, Y_L^{M-1} e^{i\varphi}\right\}.C$

In order to calculate the polarization of the emitted radiation we determine first the polarization vector $\boldsymbol{\xi}_{0L}^M$ of the radiation in direction (ϑ) in the case of (L, M)-pole radiation. Now we need the field strengths $\mathbf{E}$ and $\mathbf{B}$, which follow with (49) and (50) from the expressions for $\mathbf{A}$ given in [4]) table I, for both electric and magnetic (L, M)-pole radiation. We give the components E_r, E_ϑ, E_φ and B_r, B_ϑ, B_φ in our table I for large distance from the origin ($kr \gg 1$) (Fig. 1

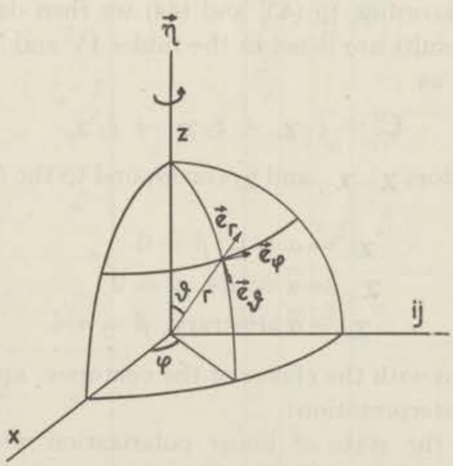

Fig. 1. The coordinates r, ϑ, φ and the axis of quantization $\boldsymbol{\eta}$. $\mathbf{e}_r$, $\mathbf{e}_\vartheta$, $\mathbf{e}_\varphi$, are unit vectors.

TABLE II

Dipole radiation					
electric		magnetic		$L = 1$	
				$M = 0$	$M = \pm 1$
$-E_\varphi$	B_ϑ	E_ϑ	B_φ	0	$\frac{1}{\sqrt{4\pi}} \frac{\sqrt{3}}{2} e^{\pm i\varphi} . C$
E_ϑ	B_φ	E_φ	$-B_\vartheta$	$-\frac{i}{\sqrt{4\pi}} \sqrt{\tfrac{3}{2}} \sin\vartheta . C$	$\pm \frac{i}{\sqrt{4\pi}} \frac{\sqrt{3}}{2} \cos\vartheta\, e^{\pm i\varphi} . C$

illustrates the coordinate system chosen). The tables II and III give these results in a more explicit form for the dipole and quadrupole case ($L = 1, L = 2$).

For the calculation of $\boldsymbol{\xi}^M_{0L}$ (for radiation in a direction given by (ϑ, φ)) we take the two unit vectors $\mathbf{e}_1$, $\mathbf{e}_2$ defined by

$$\mathbf{e}_1 = \mathbf{e}_\vartheta,\ \mathbf{e}_2 = \mathbf{e}_\varphi. \tag{63}$$

Now $\mathbf{E}$ can be written as

$$\mathbf{E} = E_\vartheta\, \mathbf{e}_1 + E_\varphi\, \mathbf{e}_2, \tag{64}$$

and (compare (36))

$$E_\varphi / E_\vartheta = c_2 / c_1 \tag{65}$$

c_2/c_1 can be calculated with the aid of Table II and III (for $L = 1$ and $L = 2$). According to (43) and (44) we then determine α, tg β and $\boldsymbol{\xi}^M_{0L}$. The results are listed in the tables IV and V.

We write $\boldsymbol{\xi}^M_{0L}$ as

$$\boldsymbol{\xi}^M_{0L} = \xi_1\, \boldsymbol{\chi}_{\parallel} + \xi_2\, \boldsymbol{\chi}_{\perp} + \xi_3\, \boldsymbol{\chi}_c. \tag{66}$$

The unit vectors $\boldsymbol{\chi}_{\parallel}$, $\boldsymbol{\chi}_{\perp}$ and $\boldsymbol{\chi}_c$ correspond to the following values of α and β

$$\begin{aligned} \boldsymbol{\chi}_{\parallel} &\to \alpha = 0,\ \beta = 0 \\ \boldsymbol{\chi}_{\perp} &\to \alpha = \pi/4,\ \beta = 0 \\ \boldsymbol{\chi}_c &\to \alpha \text{ arbitrary},\ \beta = \pi/4. \end{aligned} \tag{67}$$

In connection with the choice of the vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ this allows the following interpretation:

$\boldsymbol{\chi}_{\parallel}$ determines the state of linear polarization when the electric vector lies in the plane of $\mathbf{k}$ (direction of propagation) and $\boldsymbol{\eta}$ (axis of rotational symmetry).

TABLE III

Quadrupole radiation						
electric		magnetic		$L = 2$		
				$M = 0$	$M = \pm 1$	$M = \pm 2$
$-E_\varphi$	B_ϑ	E_ϑ	B_φ	0	$\frac{1}{\sqrt{4\pi}} \frac{\sqrt{5}}{2} \cos\vartheta\, e^{\pm i\varphi}.C$	$\pm \frac{1}{\sqrt{4\pi}} \frac{\sqrt{5}}{2} \sin\vartheta\, e^{\pm 2i\varphi}.C$
E_ϑ	B_φ	E_φ	$-B_\vartheta$	$-\frac{i}{\sqrt{4\pi}} \sqrt{\frac{15}{2}} \cos\vartheta \sin\vartheta.C$	$\mp \frac{i}{\sqrt{4\pi}} \frac{\sqrt{5}}{2} (1 - 2\cos^2\vartheta)\, e^{\pm i\varphi}.C$	$\frac{i}{\sqrt{4\pi}} \frac{\sqrt{5}}{2} \sin\vartheta \cos\vartheta\, e^{\pm 2i\varphi}.C$

TABLE V

State of polarization of quadrupole radiation									
$L = 2$	$M = 0$			$M = \pm 1$			$M = \pm 2$		
	α	$\mathrm{tg}\,\beta$	ξ_{0L}^{M}	α	$\mathrm{tg}\,\beta$	ξ_{0L}^{M}	α	$\mathrm{tg}\,\beta$	ξ_{0L}^{M}
electric	0	0	$\chi_{\|}$	0	$\mp \frac{\cos\vartheta}{1 - 2\cos^2\vartheta}$	$\frac{1 - 5\cos^2\vartheta + 4\cos^4\vartheta}{1 - 3\cos^2\vartheta + 4\cos^4\vartheta}\chi_{\|} \pm \frac{-2\cos\vartheta + 4\cos^3\vartheta}{1 - 3\cos^2\vartheta + 4\cos^4\vartheta}\chi_c$	0	$\pm \frac{1}{\cos\vartheta}$	$-\frac{1 - \cos^2\vartheta}{1 + \cos^2\vartheta}\chi_{\|} \pm \frac{2\cos\vartheta}{1 + \cos^2\vartheta}\chi_c$
magnetic	0	∞	$-\chi_{\|}$	0	$\mp \frac{1 - 2\cos^2\vartheta}{\cos\vartheta}$	$-\frac{1 - 5\cos^2\vartheta + 4\cos^4\vartheta}{1 - 3\cos^2\vartheta + 4\cos^4\vartheta}\chi_{\|} \pm \frac{-2\cos\vartheta + 4\cos^3\vartheta}{1 - 3\cos^2\vartheta + 4\cos^4\vartheta}\chi_c$	0	$\pm \cos\vartheta$	$\frac{1 - \cos^2\vartheta}{1 + \cos^2\vartheta}\chi_{\|} \pm \frac{2\cos\vartheta}{1 + \cos^2\vartheta}\chi_c$

TABLE IV

State of polarization of dipole radiation						
$L = 1$	$M = 0$			$M = \pm 1$		
	α	$\mathrm{tg}\,\beta$	$\boldsymbol{\xi}_{0L}^M$	α	$\mathrm{tg}\,\beta$	$\boldsymbol{\xi}_{0L}^M$
electric	0	0	$\boldsymbol{\chi}_{\|}$	0	$\pm \dfrac{1}{\cos\vartheta}$	$-\dfrac{1-\cos^2\vartheta}{1+\cos^2\vartheta}\boldsymbol{\chi}_{\|} \pm \dfrac{2\cos\vartheta}{1+\cos^2\vartheta}\boldsymbol{\chi}_c$
magnetic	0	∞	$-\boldsymbol{\chi}_{\|}$	0	$\pm\cos\vartheta$	$\dfrac{1-\cos^2\vartheta}{1+\cos^2\vartheta}\boldsymbol{\chi}_{\|} \pm \dfrac{2\cos\vartheta}{1+\cos^2\vartheta}\boldsymbol{\chi}_c$

$-\boldsymbol{\chi}_{\|}$ gives the state of linear polarization rotated through $\pi/2$ compared with the former.

$\pm\boldsymbol{\chi}_{\perp}$ give the states of linear polarization rotated through $\pi/4$ compared with $\boldsymbol{\chi}_{\|}$.

$+\boldsymbol{\chi}_c$, $-\boldsymbol{\chi}_c$ give left and right circular polarized radiation respectively.

The probability $F_L^M(\vartheta, \boldsymbol{\xi})$ of finding a photon with polarization vector $\boldsymbol{\xi}$ in a direction ϑ follows from (55) and (48)

$$F_L^M(\vartheta, \boldsymbol{\xi}) = F_L^M(\vartheta) \cdot \tfrac{1}{2}(1 + \boldsymbol{\xi} \cdot \boldsymbol{\xi}_{0L}^M) = 4\pi^2 r^2 \mathbf{E} \cdot \mathbf{E}^* \tfrac{1}{2}(1 + \boldsymbol{\xi} \cdot \boldsymbol{\xi}_{0L}^M). \quad (68)$$

If we now consider the ensemble of oriented radioactive nuclei, and an axis $\boldsymbol{\eta}$ of rotational symmetry exists, we can calculate in a way similar to (59) the probability $W(\vartheta, \boldsymbol{\xi})$ of finding a γ-quantum in the direction ϑ with polarization vector $\boldsymbol{\xi}$

$$W(\vartheta, \boldsymbol{\xi}) = \Sigma_{m_i M}\, a_{m_i}\, G^{j_i L j_f}_{m_i, m_i - M}\, F_L^M(\vartheta, \boldsymbol{\xi}). \quad (69)$$

We can compute $W(\vartheta, \boldsymbol{\xi})$ as $F_L^M(\vartheta)$ for ($L = 1$ and $L = 2$) is given by (57) and (58), and $\boldsymbol{\xi}_{0L}^M$ is listed in the tables IV and V.

As is clear from the formula (68) for $F_L^M(\vartheta, \boldsymbol{\xi})$ the radiation of a pure (L, M)-pole is totally polarized. The radiation from an ensemble of oriented nuclei, however, is in general partially polarized. $W(\vartheta, \boldsymbol{\xi})$ can also be written as (47)

$$W(\vartheta, \boldsymbol{\xi}) = W(\vartheta) \cdot \tfrac{1}{2}(1 + P\boldsymbol{\xi} \cdot \boldsymbol{\xi}_0). \quad (70)$$

As $W(\vartheta)$ is known from (59), and $W(\vartheta, \boldsymbol{\xi})$ from (69), we can obtain P and $\boldsymbol{\xi}_0$ from (70). For the calculation of the degree of polarization P and the polarization vector $\boldsymbol{\xi}_0$ it is sufficient to know $W(\vartheta)\, P\boldsymbol{\xi}_0$. Therefore we give the results for $W(\vartheta)\, P\boldsymbol{\xi}_0$, instead of for $W(\vartheta, \boldsymbol{\xi})$ in § 7.

From (59), (68), (69) and (70) it follows that

$$W(\vartheta)\, P\boldsymbol{\xi}_0 = \Sigma_{m_i M}\, a_{m_i}\, G^{j_i L j_f}_{m_i m_i - M}\, F_L^M(\vartheta)\, \boldsymbol{\xi}_{0L}^M. \quad (71)$$

§ 5. *Another method of calculation of the angular distribution and polarization of 2^L-pole γ-radiation.* In this section we shall derive formulae for $W(\vartheta)$ and $W(\vartheta)\, P\xi_0$ for arbitrary 2^L-pole γ-radiation, making use of the algebra of tensor operators developed by Racah [10]) [6]). During the derivation we shall drop constant factors without further comment since these factors only affect the normalization which is not essential in the problem. If $\boldsymbol{\eta}$ is the axis of rotational symmetry and of quantization, then the probability of emission of a γ-quantum in the direction **k** with polarization described by **c** (§ 3) is given by

$$W(\mathbf{k}, \mathbf{c}, \boldsymbol{\eta}) = \Sigma_{m_i m_f}\, a_{m_i}\, |\langle j_i\, m_i|\, H|j_f\, m_f\rangle|^2. \tag{72}$$

In the case of a pure 2^L-pole radiation we can write for the interaction Hamiltonian $\bar{H}$, in a coordinate system with **k** as quantization axis,

$$\bar{H} = \Sigma_M\, a_{LM}(\mathbf{c})\, T^L_M. \tag{73}$$

T^L_M are the components of an irreducible tensor operator of degree L, which operate on the nucleus. $a_{LM}(\mathbf{c})$ are functions of **c**, and are thus connected with the polarization of the emitted γ-quantum. Only a_{L1}, and a_{L-1} are different from zero and are expressed by **c** as follows [8]), [6]).

$$a_{L1}=-(c_1-ic_2)/\sqrt{2},\ a_{L-1}=(c_1+ic_2)/\sqrt{2}\ \text{(electric } 2^L\text{-pole radiation)} \tag{74a}$$

$$a_{L1}=-(c_2+ic_1)/\sqrt{2},\ a_{L-1}=(c_2-ic_1)/\sqrt{2}\ \text{(magnetic } 2^L\text{-pole radiation)} \tag{74b}$$

With $\boldsymbol{\eta}$ as quantization axis the Hamiltonian (73) takes the form (with $(\mathbf{k}\cdot\boldsymbol{\eta}) = \cos\vartheta$; cf. [6]) formula (2))

$$H = \Sigma_{M\mu}\, a_{LM}(\mathbf{c})\, T^L_\mu D^L_{\mu M}\,(0, \vartheta, 0). \tag{75}$$

$(0, \vartheta, 0)$ are the three Euler angles associated with the rotation which transforms the coordinate system with **k** as z-axis into that whose z-axis is $\boldsymbol{\eta}$. Making use of (72) and (75) we find for $W(\mathbf{k}, \mathbf{c}, \boldsymbol{\eta})$

$$W(\vartheta, \mathbf{c}) = W(\mathbf{k}, \mathbf{c}, \boldsymbol{\eta}) =$$

$$=\Sigma_{m_i m_f M M' \mu\mu'}\, a_{m_i}\, a^*_{LM} a_{LM'} \langle j_i m_i\, |T^L_\mu|\, j_f m_f\rangle^* \langle j_i m_i| T^L_{\mu'}|\, j_f m_f\rangle\, D^{L*}_{\mu M} D^L_{\mu' M'}. \tag{76}$$

With the relation [19]) page 203, (16a))

$$D^{L*}_{\mu M} D^L_{\mu' M'} = \Sigma_{\varrho\sigma k}(-1)^{M-\mu} \langle L{-}\mu L\mu'\, |LLk\varrho\rangle\, D^k_{\varrho\sigma} \langle LL\, k\sigma|\, L{-}MLM'\rangle \tag{77}$$

and the formulae, (16′) and (29) resp., from [10])

$$\langle j_i m_i | T^L_\mu | j_f m_f \rangle = (-1)^{j_i + m_i} \langle j_i || T^L || j_f \rangle \cdot V(j_i j_f L; -m_i m_f \mu), \quad (78)$$

$$\langle L - \mu L \mu' | LL\, k\varrho \rangle = (-1)^{k+\varrho} (2k+1)^{\frac{1}{2}} V(LLk; -\mu\mu' - \varrho), \quad (79)$$

formula (76) becomes (with summation over m_i, m_f, M, M', μ, μ', k, ϱ and σ)

$$W(\vartheta, \mathbf{c}) = \Sigma\, a_{m_i} a^*_{LM} a_{LM'} (-1)^{k+\varrho+M-\mu} (2k+1)^{\frac{1}{2}} D^k_{\varrho\sigma} \times$$
$$\times \langle LLk\sigma \mid L - MLM' \rangle V(j_i j_f L; -m_i m_f \mu) V(j_i j_f L; -m_i m_f \mu') \times$$
$$\times V(LLk; -\mu\mu'\varrho). \quad (80)$$

With formula (41) from [10]) the summation over μ, μ' and m_f can be carried out and $W(\vartheta, \mathbf{c})$ becomes

$$W(\vartheta, \mathbf{c}) = \Sigma\, a_{m_i} a^*_{LM} a_{LM'} (-1)^{M-L} D^k_{\varrho\sigma} \langle LLk\sigma \mid L - MLM' \rangle \times$$
$$\times W(j_f j_i Lk; L_{j_i}) (-1)^{j_i - m_i} (2k+1)^{\frac{1}{2}} (-1)^{k-\varrho} V(j_i j_i k; -m_i m_i \varrho). \quad (81)$$

Now we observe that (cf. formula (17) from [10])) that $V(j_i j_i k; -m_i m_i \varrho) = V(j_i j_i k; -m_i m_i 0)\, \delta_{0\varrho}$, which makes possible summation over ϱ. From this we obtain with the aid of (79), (25), (26) and (32)

$$W(\vartheta, \mathbf{c}) = \Sigma_{k\sigma} (-1)^k C_{k\sigma}(LL)\, W(j_f j_i Lk; Lj_i)\, f_k w_k^{-1} D^k_{0\sigma}(0, \vartheta, 0), \quad (82)$$

where $C_{k\sigma}(LL)$ is the abbreviation

$$C_{k\sigma}(LL) = \Sigma_{MM'} (-1)^{L-M} a^*_{LM} a_{LM'} \langle LL\, k\sigma \mid L - MLM' \rangle. \quad (83)$$

We shall now consider electric 2^L-pole radiation by substituting (74a) in (83). Then only $C_{k0}(LL)$ and $C_{k\pm 2}(LL)$ turn out to be different from zero. Making use of (45) we find

$$C_{k0}(LL) = (-1)^{L-1} \langle LLk0 \mid L1L - 1 \rangle \text{ if } k \text{ is even}, \quad (84a)$$

$$C_{k0}(LL) = -(-1)^{L-1} \langle LLk0 \mid L1L - 1 \rangle\, \xi_3, \text{ if } k \text{ is odd}, \quad (84b)$$

$$C_{k2}(LL) + C_{k-2}(LL) = (-1)^{L-1} \langle LLk2 | L1L1 \rangle . -\xi_1 \text{ if } k \text{ is even}, \quad (84c)$$

$$C_{k2}(LL) + C_{k-2}(LL) = 0 \text{ if } k \text{ is odd}. \quad (84d)$$

With (84) and $D^k_{0\sigma} = Y^\sigma_k(\vartheta, 0)\, (2k+1)^{-\frac{1}{2}}$, (82) becomes

$$W(\vartheta, \boldsymbol{\xi}) = W(\vartheta, \mathbf{c}) =$$
$$= \Sigma_{k\ even} W(j_f j_i Lk; Lj_i)\, f_k w_k^{-1} (2k+1)^{-\frac{1}{2}} \{ Y^0_k \langle LLk0 \mid L1L - 1 \rangle -$$
$$- Y^2_k \langle LLk2 | L1L1 \rangle \cdot \xi_1 \} +$$
$$+ \Sigma_{k\ odd} W(j_f j_i Lk; Lj_i)\, f_k w_k^{-1} (2k+1)^{-\frac{1}{2}} Y^0_k \langle LLk0 | L1L - 1 \rangle \cdot \xi_3. \quad (85)$$

If the polarization is not observed, we can sum over the polarization directions and obtain $W(\vartheta) = W(\vartheta, \boldsymbol{\xi}) + W(\vartheta, -\boldsymbol{\xi})$. From (85)

$$W(\vartheta) =$$
$$= \Sigma_{k\,even}\, 2W(j_f j_i Lk; Lj_i)\, f_k w_k^{-1} (2k+1)^{-\frac{1}{2}}\, Y_k^0(\vartheta, 0) \langle LLk0|L1L-1\rangle. \quad (86)$$

Making use of (86) and of the representation

$$\boldsymbol{\xi} = \xi_1\, \boldsymbol{\chi}_{||} + \xi_2\, \boldsymbol{\chi}_{\perp} + \xi_3\, \boldsymbol{\chi}_c, \quad (87)$$

it follows that (85) can be written

$$W(\vartheta, \boldsymbol{\xi}) = \tfrac{1}{2} W(\vartheta)\, (1 + P\boldsymbol{\xi}_0 \cdot \boldsymbol{\xi}), \quad (88)$$

with

$$W(\vartheta)\, P\boldsymbol{\xi}_0 =$$
$$= \Sigma_{k\,odd}\, 2W(j_f j_i Lk; Lj_i)\, f_k w_k^{-1} (2k+1)^{-\frac{1}{2}}\, Y_k^0(\vartheta, 0)\, \langle LLk0|L1L-1\rangle\, \boldsymbol{\chi}_c -$$
$$- \Sigma_{k\,even}\, 2W(j_f j_i Lk; Lj_i)\, f_k w_k^{-1} (2k+1)^{-\frac{1}{2}} Y_k^2(\vartheta, 0)\, \langle LLk2|L1L1\rangle\, \boldsymbol{\chi}_{||}. \quad (89)$$

For magnetic 2^L-pole radiation we must replace $\boldsymbol{\chi}_{||}$ by $-\boldsymbol{\chi}_{||}$ in (89), as is easily derived from (74*b*) in the same way as in the electric case. Formula (86) remains unchanged.

$W(\vartheta)$ in formula (86) is not normalized according to (62), but this normalization can be obtained by calculating $\int W(\vartheta)\, d\Omega$.

§ 6. *Explicit formulae for the angular distribution of γ-radiation emitted by oriented nuclei.* We have calculated explicit formulae for the angular distribution function $W(\vartheta)$ (59) and the polarization $W(\vartheta)\, P\boldsymbol{\xi}_0$ (71) for the following cases:

dipole radiation $j_f = j_i \pm 1$, $j_f = j_i$,
quadrupole radiation $j_f = j_i \pm 2$.

j_i and j_f are the angular momentum quantum numbers of the initial and final nuclei. We have assumed that there are no interference effects among radiations of different multipole character. The formulae are then valid for pure dipole or quadrupole radiation. We have made the calculations by methods both of § 4 and of § 5. The method of § 4 does not give rise to long calculations in the dipole case, but for the next order, quadrupole radiation, the amount of labour required already begins to mount. For higher multipole orders the method of § 5 is certainly to be preferred. The computing work to get explicit formulae with the aid of the latter method is now shifted to the evaluation of the general expressions in terms of simple products, etc. Especially, the work to compute the

$W(j_f j_i Lk, L j_i)$ is considerable. However, once they have been calculated for a number of cases they can be tabulated and can be used for many purposes (tables have been given by K. Alder [8]).

The results for $W(\vartheta)$ are the same for electric and magnetic 2^L-pole radiation. The polarization, however, is different for the electric and magnetic radiations (§ 7). We give here explicit formulae for $W(\vartheta)$ (normalized according to (62)).

Dipole radiation ($L = 1$).

$$j_f = j_i - 1, \quad W(\vartheta) = 2\,(1 + \tfrac{3}{2}N_2 f_2 P_2(\cos\vartheta)) \tag{90}$$

$$j_f = j_i \quad , \quad W(\vartheta) = 2\,(1 - \tfrac{3}{2}K_2 f_2 P_2(\cos(\vartheta)) \tag{91}$$

$$j_f = j_i + 1, \quad W(\vartheta) = 2\,(1 + \tfrac{3}{2}M_2 f_2 P_2(\cos\vartheta)) \tag{92}$$

Quadrupole radiation ($L = 2$)

$$j_f = j_i - 2, \quad W(\vartheta) = 2\,(1 - \tfrac{15}{7}N_2 f_2 P_2(\cos\vartheta) - 5N_4 f_4 P_4(\cos\vartheta)) \tag{93}$$

$$j_f = j_i + 2, \quad W(\vartheta) = 2(1 - \tfrac{15}{7}M_2 f_2 P_2(\cos\vartheta) - 5M_4 f_4 P_4(\cos\vartheta)) \tag{94}$$

with

f_k the degree of orientation of order k ((16), (18)),

$$P_2(\cos\vartheta) = \tfrac{3}{2}(\cos^2\vartheta - \tfrac{1}{3}) \tag{95a}$$

$$P_4(\cos\vartheta) = \tfrac{35}{8}(\cos^4\vartheta - \tfrac{6}{7}\cos^2\vartheta + \tfrac{3}{35}) \tag{95b}$$

$$N_k = b_k\, j^k\,(2j-k)!/(2j)!, \quad M_k = b_k\, j^k\,(2j+1)!/(2j+k+1)! \tag{96a}$$

$$b_k = 2^{\frac{1}{2}k} \text{ if } k \text{ is even}, \quad b_k = 2^{\frac{1}{2}(k+1)} \text{ if } k \text{ is odd}, \tag{96b}$$

$$K_1 = 1/(j_i+1), \quad K_2 = j_i/(j_i+1). \tag{97}$$

§ 7. *Explicit formulae for the polarization of γ-radiation emitted by oriented nuclei.* We now give explicit formulae for $W(\vartheta)\, P\xi_0$. The formulae given below are valid for *electric* dipole and quadrupole radiation. For the magnetic radiation, the sign of $\chi_{\|}$ is changed in the corresponding formulae for the electric case, while the sign of χ_c remains unaltered.

Electric dipole radiation ($L = 1$)

$$j_f = j_i - 1, \quad W(\vartheta)\, P\xi_0 = 3N_1 f_1 \cos\vartheta\, \chi_c - \tfrac{9}{2}N_2 f_2\,(1 - \cos^2\vartheta)\, \chi_{\|} \tag{98}$$

$$j_f = j_i \quad , \quad W(\vartheta)\, P\xi_0 = 3K_1 f_1 \cos\vartheta\, \chi_c + \tfrac{9}{2}K_2 f_2\,(1 - \cos^2\vartheta)\, \chi_{\|} \tag{99}$$

$$j_f = j_i + 1, \quad W(\vartheta)\, P\xi_0 = -3M_1 f_1 \cos\vartheta\, \chi_c - \tfrac{9}{2}M_2 f_2\,(1 - \cos^2\vartheta)\, \chi_{\|} \tag{100}$$

Electric quadrupole radiation ($L = 2$)

$j_f = j_i - 2$

$$W(\vartheta)\, P\xi_0 = \{\tfrac{45}{7} N_2 f_2 (\cos^2\vartheta - 1) + \tfrac{25}{4} N_4 f_4 (-7\cos^4\vartheta + 8\cos^2\vartheta - 1)\}\, \chi_{\parallel} + \\ + \{2N_1 f_1 \cos\vartheta + 5N_3 f_3 (-5\cos^3\vartheta + 3\cos\vartheta)\}\, \chi_c. \tag{101}$$

$j_f = j_i + 2$

$$W(\vartheta)\, P\xi_0 = \{\tfrac{45}{7} M_2 f_2 (\cos^2\vartheta - 1) + \tfrac{25}{4} M_4 f_4 (-7\cos^4\vartheta + 8\cos^2\vartheta - 1)\}\, \chi_{\parallel} - \\ - \{2M_1 f_1 \cos\vartheta + 5\, M_3 f_3 (-5\cos^3\vartheta + 3\cos\vartheta)\}\, \chi_c. \tag{102}$$

The angular dependent functions which occur in (98).... (102) are proportional to $Y_k^q(\vartheta, 0)$ as also follows from (89). Here f_k, N_k, M_k, K_1 and K_2 have the same meaning as in § 6.

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Chapter II

GAMMA RADIATION EMITTED BY ORIENTED NUCLEI

THE INFLUENCE OF PRECEDING RADIATIONS; THE EVALUATION OF EXPERIMENTAL DATA

Synopsis

The angular distribution and polarization of γ-radiation emitted by oriented nuclei was expressed earlier [1]) with the aid of parameters f_k. These parameters characterize the state of orientation of the nuclei from which the radiation is emitted. Here explicit formulae are derived for the change of the parameters f_k if the γ-radiation under consideration is preceded by a β or a γ-transition.

A discussion is given of the data of physical interest, which may be obtained by the analysis of experimental data on γ-radiation from oriented nuclei: multipole character of γ-transitions, nuclear spins and parities, nuclear magnetic moments, data on the Hamiltonian for the β-interaction and nuclear matrix-elements for β-decay.

§ 1. *Introduction.* For experiments in which radioactive nuclei are oriented the life time of the nuclei must be rather large. Most half lifes for γ-transitions are very short, except for the isomeric transitions. However, there are no, or very few, isomeric nuclei which have a suitable half life, a suitable energy (in addition the γ-radiation must not be entirely converted) and which can be oriented by the present experimental methods. Hence, the situation for observation of the angular distribution and polarization of the γ-radiation emitted by oriented nuclei will generally be the following:

a) the nuclei which are oriented will be β-radioactive with a sufficiently large life-time.

b) the β-transition is followed by a γ-transition (fig. 1) or possibly by two or more γ-transitions in cascade (fig. 2).

For the calculation of the angular distribution and polarization of the γ-radiation, the formulae of I, §§ 6–7 [1]) apply. We must only keep in mind that in a β-transition $j_0 \to j_i$ according to fig. 1 the initial orientation of the nuclei with angular momentum j_0 is disturbed. If we consider the distribution of the second γ-radiation according to fig. 2 we must also account for the change of orientation caused by the emission of the first γ-radiation.

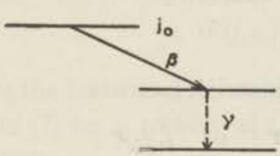

Fig. 1. Decay scheme.

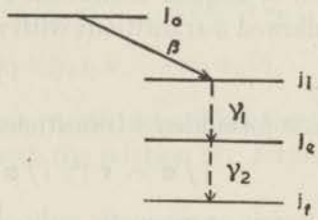

Fig. 2. Decay scheme.

In the next section we shall derive formulae connecting the initial orientation with the orientation after the β or γ-transitions, which precede the observed γ-radiation. These results together with the formulae of I provide the theoretical formulae, necessary for the discussion of experimental data in this field. Which data of physical interest can be obtained will be discussed in §§ 3 and 4.

§ 2. *Calculation of the change of the orientation parameters f_k by β or γ-transitions.* We assume that there is an axis η of rotational symmetry. Then (cf. I, § 2) the orientation of the nuclei with spin j_0 is completely characterized by the relative populations a_{m_0} of the sublevels m_0 ($\Sigma_{m_0=-j_0}^{j_0} a_{m_0} = 1$). After a β-transition (or a γ-transition) $j_0 \to j_i$ the probabilities a_{m_i} are connected with a_{m_0} by (cf. I, § I formulae 3, 4)

$$a_{m_i} = \Sigma_{m_0} a_{m_0} P_{m_0 m_i}, \qquad (1)$$

$$\Sigma_{m_i} P_{m_0 m_i} = 1, \qquad (2)$$

where $P_{m_0 m_i}$ is the partial transition probability for the transition $(j_0 m_0) \to (j_i m_i)$. For special cases $P_{m_0 m_i}$ are given by the following expressions.

A. $$P^{(0)}_{m_0 m_i} = \delta_{m_0 m_i} \qquad (3)$$

This formula is valid for

a) allowed β-transitions with matrix elements

$$|\textstyle\int 1|^2 \text{ or } |\textstyle\int \gamma_5|^2$$

b) a forbidden β-transition with a scalar matrix element.

In these two cases there will often be other matrix elements which play a role in $P_{m_0 m_i}$

B. $$P^{(1)}_{m_0 m_i} = |\langle j_i m_i\, 1M |\, j_i\, 1 j_0 m_0 \rangle|^2 \tag{4}$$

Here $\langle j_i m_i\, 1M |\, j_i\, 1 j_0 m_0\rangle$ are the transformation coefficients for the addition of angular momenta. Formula (4) applies for

a) allowed β-transitions with matrix element

$$|\int \boldsymbol{\sigma}\,|^2$$

b) first forbidden β-transitions with matrix elements

$$|\int \boldsymbol{\sigma} \wedge \mathbf{r}\,|^2,\ |\int \boldsymbol{\alpha}\,|^2,\ |\int \mathbf{r}\,|^2 \text{ or } |\int \gamma_5\, \mathbf{r}\,|^2$$

c) electric or magnetic γ-dipole transitions.

C. $$P^{(2)}_{m_0 m_i} = |\langle j_i m_i\, 2\, M \,|\, j_i \,|\, 2 j_0\, m_0 \rangle|^2 \tag{5}$$

This formula applies for

a) first forbidden β-transitions with matrix element

$$B_{ij}$$

b) second forbidden β-transitions with matrix elements

$$R_{ij},\ A_{ij},\ T_{ij},\ \text{or}\ R^{\gamma}_{ij}$$

c) electric or magnetic γ-quadrupole transitions.

We shall consider a transition for which the formula

$$P^{(L)}_{m_0 m_i} = |\langle j_i\, m_i\, LM \,|\, j_i\, L j_0\, m_0 \rangle|^2 \tag{6}$$

is valid, of which the above mentioned cases are examples. Since from I, §§ 6–7 it is clear that for the description of the orientation only the parameters f_k are needed, we derive a relation between $f_k(j_0)$ giving the orientation before the transition and $f_k(j_i)$ giving the orientation after the transition. We start with the formula (cf. I, § 2 formulae 25 and 26)

$$f_k(j_i) = w_k(j_i)\, \Sigma_{m_i} (-1)^{j_i - m_i} \langle j_i m_i j_i - m_i |\, j_i j_i\, k0 \rangle\, a_{m_i}. \tag{7}$$

According to (1) and (6)

$$a_{m_i} = \Sigma_{m_0}\, a_{m_0}\, |\langle j_i\, m_i\, LM \,|\, j_i L j_0\, m_0 \rangle|^2. \tag{8}$$

With Racah's definition of the functions V (cf. [2]) formula (16'))

$$\langle j_i\, m_i\, LM |\, j_i L j_0 m_0 \rangle = (-1)^{j_0 + m_0}\, (2j_0 + 1)^{1/2}\, V(j_i L j_0;\, m_i\, M - m_0), \tag{9}$$

it follows from (7) and (8) that

$$f_k(j_i) = w_k(j_i)\,\Sigma_{m_0 m_i}\, a_{m_0} (-1)^{j_i - m_i + k} (2k+1)^{\frac{1}{2}} (2j_0+1)\, V(j_i j_i k;\, m_i - m_i 0) \times$$
$$\times\, V(j_i L j_0;\, m_i M - m_0)\, V(j_i L j_0;\, m_i M - m_0). \tag{10}$$

Summation in (10) over m_i and M with formula (41) of Racah's paper [2]) gives the result

$$f_k(j_i) = w_k(j_i)\,\Sigma_{m_0}\, a_{m_0}\, (2k+1)^{\frac{1}{2}}\, (2j_0+1)\, (-1)^{L+k-j_i+m_0} \times$$
$$\times\, W(j_i j_0 j_i j_0;\, Lk)\, V(j_0 j_0 k;\, -m_0 m_0 0), \tag{11}$$

where the functions W are the Racah coefficients. With the application of (7) for j_0 instead of j_i and with the relation (9), formula (11) becomes

$$f_k(j_i) = w_k(j_i)\, w_k(j_0)^{-1}\, (2j_0+1)\, W(j_i L k j_0;\, j_0 j_i)\, f_k(j_0). \tag{12}$$

From the explicit expressions for w_k (I, § 2 formula 32) the product

$$w_k(j_i)\, w_k(j_0)^{-1} = (j_0/j_i)^k \left[\frac{(2j_i+k+1)!\,(2j_0-k)!}{(2j_i-k)!\,(2j_0+k+1)!}\right]^{\frac{1}{2}} \tag{13}$$

is obtained.

Tables for the Racah coefficient $W(j_i L k j_0;\, j_0 j_i)$ are given in several papers [3]) [4]) [5]). With (12) we can directly calculate $f_k(j_i)$ from $f_k(j_0)$ when j_0, j_i and L are known.

In some cases the relation (12) becomes very simple. For the transition $j_0 \to j_0 - L$,

$$f_k(j_i) = \frac{j_0^k\,(2j_0-k)!}{(2j_0)!}\,\frac{(2j_i)!}{j_i^k(2j_i-k)!}\, f_k(j_0) \tag{14}$$

or

$$N_k(j_i)\, f_k(j_i) = N_k(j_0)\, f_k(j_0), \tag{15}$$

where the functions $N_k(j)$ are the same as defined by I, § 6 formula (96a). For $j_0 \to j_0 + L$,

$$f_k(j_i) = \frac{j_0^k\,(2j_0+1)!}{(2j_0+k+1)!}\,\frac{(2j_i+k+1)!}{j_i^k(2j_i+1)!}\, f_k(j_0) \tag{16}$$

or

$$M_k(j_i)\, f_k(j_i) = M_k(j_0)\, f_k(j_0), \tag{17}$$

with $M_k(j)$ again defined by I, § 6 formula (96*a*).

For $j_0 \to j_0$ with total angular momentum quantum number of the emitted radiation $L = 1$ we find

$$f_k(j_i) = \left[1 - \frac{k(k+1)}{2j_0(j_0+1)}\right] f_k(j_0). \tag{18}$$

§ 3. *Example of the application of the formulae to the angular distribution of γ-radiation from oriented ⁶⁰Co nuclei.* The formulae (I, § 6–7, formulae 90–102) for the angular distribution and polarization of γ-radiation can be applied to actual cases using the results of § 2. We shall illustrate this by dealing with oriented ^{60}Co nuclei of which the radiation has been investigated experimentally by D a n i e l s et al. [6]), G o r t e r et al. [7]), G r a c e [8]) and P o p p e m a et al. [9]).

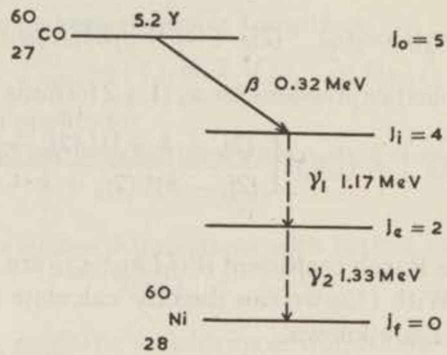

Fig. 3. Decay scheme of ^{60}Co.

We assume the decay scheme to be as is shown in fig. 3 (cf. D e u t s c h [10])) where the transitions are electric quadrupole transitions. Then I, § 6, formula 93 for the angular distribution applies i.e.

$$W(\vartheta) = 2(1 - \tfrac{15}{7} N_2(j_i)\, f_2(j_i)\, P_2(\cos\vartheta) - 5N_4(j_i)\, f_4(j_i) P_4(\cos\vartheta)). \tag{19}$$

The formula (19) is written down for the angular distribution of the first γ-radiation. Initially, the orientation parameters $f_k(j_0)$ are given and the $f_k(j_i)$ are calculated for the evaluation of (19). However, for ^{60}Co the formula (15) may be applied and we need to compute only $N_k(j_0)\, f_k(j_0)$ for $k = 2$ and 4. We have evaluated $f_k(j_0)$ for $j_0 = 5$ as a function of β with the assumption

$$a_{m_0} = C \exp. (\beta m_0) \tag{20}$$

which is valid if we have a Boltzmann distribution over equidistant energy levels (with $\beta = \mu B/kTj_0$, μ nuclear magnetic moment, B magnetic field at the place of the nucleus). The results for f_1, f_2, f_3 and f_4 are given in fig. 4 to give an idea of the magnitude of these parameters and their functional dependence on β.

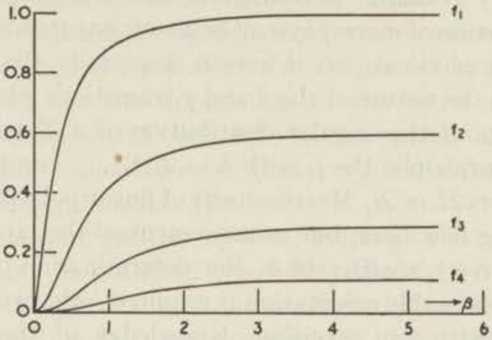

Fig. 4. Diagram of the orientation parameters f_1, f_2, f_3 and f_4 as a function of β.

For orientations of the nuclei, which differ not too much from spherical symmetry, the parameters f_k may be evaluated with the aid of approximate formulae for (20). If e.g. the probabilities a_{m_0} can be approximated by a polynomial of the fourth degree we use the formulae 20, 21 and 22 of. I, § 2.

Using the obtained values for f_k in (19) we obtain numerical results for $W(\vartheta)$. Poppema et al. [9]) compared the results with experimental data and found a rather good agreement. Experimentally, the angular distribution of the first and second γ-radiation (fig. 3) is measured together since the energy difference of the γ quanta cannot be separated easily with scintillation counters. Nevertheless the results obtained from (19) can directly be applied for the following reason. The angular distribution of the second radiation emitted by the nuclei with spin j_e is determined by (19) if j_i is replaced by j_e. But again (15) holds and

$$N_k(j_i)\, f_k(j_i) = N_k(j_e)\, f_k(j_e) \tag{21}$$

which gives the result that the first and second γ-radiation have identical distributions. This will not always be the case for two successive γ-radiations as follows from the formulae (12) and e.g. (18).

§ 4. *Data of physical interest, which may be obtained from measurements on the γ-radiation emitted by oriented nuclei.* Depending on the nucleus which is oriented, information about the orientation mechanism or the nuclear process (especially the values of the nuclear spins involved) may be known or still lacking. Generally speaking, knowledge about the unknowns may be obtained if some information is already available, so that the situation is not too complex.

1. Information of *macrophysical or atomic* nature may be obtained if the nuclear physical part is known, especially the nuclear spins involved and the nature of the β and γ transitions which occur. The measurements of the angular distribution of a 2^L-pole γ-radiation provides, in principle, the f_k with $k = 2, 4, \ldots$ up to the smallest of the numbers $2L$ or $2j_i$. Measurement of linear polarization does not give anything new here, but measurement of the circular polarization would give f_k's with odd k. For determination of the f_k's with high k, a considerable orientation is required, otherwise they cannot be obtained with any precision. Knowledge of the f_k's obtained from such experiments can be used

a) in order to obtain knowledge about the *mechanism of orientation* if this is not known, or

b) if this mechanism is known, to obtain the *temperature* from the parameters f_k, so that the angular distribution of the γ-radiation can be used as a *thermometer*.

2. Information concerning *nuclear physical data* can be acquired if the mechanism of orientation is sufficiently known. Data of the following nature may be obtained:

a) *The multipole order* of the γ-radiation may be determined from the angular distribution since $W(\vartheta)$ strongly depends on L (see e.g. I § 6). For the decision as to whether we have *magnetic* or *electric* 2^L-pole radiation the linear polarization of the γ-radiation must be measured.

b) The values of the *nuclear spins and parities* may follow from the multipole order and the electric or magnetic character of the γ-transitions. The temperature dependence of the angular distribution will also depend on j_0. If j_i and j_f are known, j_0 may be determined in this way.

c) The *magnetic moment of the initial nucleus* may be obtained if the mechanism of orientation as well as the character of the nuclear process [11]) [12]) is known. If the population of the different m-levels is characterized by a Boltzmann factor

$$a_{m_0} = \exp. [(\mu B/kTj_0)\, m_0] = \exp. (\beta m_0), \qquad (22)$$

we can determine from the measured f_k the value of β and from β the value of μ if B/kTj_0 is known. One obtains only the absolute magnitude of the magnetic moment by measuring the angular distribution (or linear polarization) of the γ-radiation. The sign could only be obtained by measuring the circular polarization.

This way of measuring magnetic moments of radioactive nuclei is of interest because most radioactive nuclei are not available in sufficient quantities for the application of the usual methods. The magnetic moments of odd-odd nuclei have a special interest because they are mostly radioactive, so that very few data on magnetic moments of odd-odd nuclei exist.

d) If the spins and parities involved in the γ-radiation are known, *information* may be obtained *about the preceding β-transition*. This will be particularly true if we have an allowed β transition for which $j_0 = j_i$. The function $P_{m_0 m_i}$ (occurring in (1)) giving the partial transition probability is

$$P_{m_0 m_i} = \lambda P^{(0)}_{m_0 m_i} + \lambda' P^{(1)}_{m_0 m_i} \text{ (with } \lambda + \lambda' = 1) \tag{23}$$

for an allowed β-transition, where $P^{(0)}_{m_0 m_i}$ and $P^{(1)}_{m_0 m_i}$ are given by (3) and (5). λ and λ' are determined by

$$\lambda'/\lambda = [(c_3^2 + c_4^2)/(c_1^2 + c_2^2)]\,[|\int \boldsymbol{\sigma}|^2/|\int 1|^2]. \tag{24}$$

We refer for these notions on β-decay to[13]). $|\int \boldsymbol{\sigma}|^2$ and $|\int 1|^2$ are nuclear matrix elements for β-decay. If λ could be measured and if the ratio of the matrix elements were known one could determine the quantity $(c_3^2 + c_4^2)/(c_1^2 + c_2^2)$, which gives the relative magnitude of Gamow-Teller and Fermi terms in the Hamiltonian for the β-interaction. If on the other hand, the value of $(c_3^2 + c_4^2)/(c_1^2 + c_2^2)$ were known, one could calculate the ratio of the matrix elements $|\int \boldsymbol{\sigma}|^2$ and $|\int 1|^2$.

We may indicate in somewhat more detail how λ could be determined if we measure $f_2(j_i)$ and $f_4(j_i)$ in case of a quadrupole γ-transition preceded by a β-transition with $j_0 = j_i$. The connection of $f_2(j_i)$ and $f_4(j_i)$ with $f_2(j_0)$ and $f_4(j_0)$ is given according to (3), (18) and (23) by

$$f_2(j_i) = \lambda f_2(j_0) + (1 - \lambda)\cdot[(j_0^2 + j_0 - 3)/j_0(j_0 + 1)]\, f_2(j_0), \tag{25}$$

$$f_4(j_i) = \lambda f_4(j_0) + (1 - \lambda)\,[(j_0^2 + j_0 - 10)/j_0(j_0 + 1)]\, f_4(j_0). \tag{26}$$

We now suppose that the mechanism of orientation is known, then we may assume that $f_4(j_0)$ is a known function of $f_2(j_0)$:

$$f_4(j_0) = F\{f_2(j_0)\}. \tag{27}$$

If $f_2(j_i)$ and $f_4(j_i)$ are measured, the 3 unknowns $f_2(j_0)$, $f_4(j_0)$ and λ may be solved from the 3 equations (25), (26), (27). By making the measurements for a number of temperatures, one could reach a higher precision for λ.

A situation like this is realized for ^{58}Co for which $j_0 = j_i = 2$; $j_f = 0$. Probably this is an allowed unfavoured transition with $f_{5/2}$-$p_{3/2}$ orbitals for the odd nucleons in ^{58}Co. (25) and (26) reduce to

$$f_2(j_i) = \tfrac{1}{2}(1 + \lambda)\, f_2(j_0), \tag{28}$$

$$f_4(j_i) = \tfrac{1}{3}(-\,2 + 5\lambda)\, f_4(j_0). \tag{29}$$

This means that

$$f_4(j_i)/f_2(j_i) = -\,(\tfrac{4}{3})\, f_4(j_0)/f_2(j_0) \text{ for } \lambda = 0, \tag{30}$$

$$f_4(j_i)/f_2(j_i) = f_4(j_0)/f_2(j_0) \text{ for } \lambda = 1. \tag{31}$$

The strong dependence of $f_4(j_i)/f_2(j_i)$ on λ means that λ can probably be determined with reasonable accuracy from such measurements.

A determination of $|\int \boldsymbol{\sigma}|^2/|\int 1|^2$ for ^{58}Co would be of interest to test theories of the nuclear matrix elements for odd-odd nuclei such as proposed by B r y s k [14]).

We may add the remark that in all these considerations we assume the nuclei to have no appreciable spin precession after the β transition and before the γ-transition. In the case of ^{58}Co we have a disintegration by β^+ transition or by K capture. Since the larger part of the disintegration occurs by K capture it is possible that our assumption does not hold. The disappearance of the K electron may produce a magnetic field strong enough to cause an appreciable spin precession.

In addition to their use for the study of the γ-radiation itself, sources with oriented nuclei might eventually be suitable sources of linearly or circularly polarized γ-radiation, which could be used in other experiments.

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Chapter III

ANGULAR DISTRIBUTION OF RADIATION EMITTED BY ARBITRARY ENSEMBLES OF NUCLEI

Synopsis

An arbitrary ensemble is described by a density matrix ϱ with arbitrary values of its matrix elements. An equivalent description is given by Fano's statistical tensors, which are used in this chapter. A simple closed formula for the angular distribution of nuclear radiation from an arbitrary ensemble is derived.

§ 1. *Introduction.* An arbitrary ensemble of nuclei must be described by a density matrix,

$$\varrho_{mm'} = \overline{\overline{c^*_{m'}\, c_m}}\,, \tag{1}$$

where the double bar indicates an ensemble average. The numbers c_m are the coefficients of an expansion of a pure state $|A\rangle$ into basic vectors $|m\rangle$

$$|A\rangle = \Sigma_m\, c_m|m\rangle. \tag{2}$$

Here m is the magnetic quantum number; the angular momentum of all nuclei is supposed to be j. In this chapter we shall investigate the angular distribution of nuclear radiation emitted by an arbitrary ensemble of nuclei. This problem presents itself in the theoretical treatment of many phenomena as will be illustrated by the following considerations.

First of all, if the quantization axis has rotational symmetry a simplification is obtained since in this case the density matrix is on diagonal form. A first example of this situation is found in the problem of the angular distribution of radiation in a nuclear reaction caused by unidirectional irradiation (which direction is taken as quantization axis). A second example is the directional correlation of successive radiations, when the direction of the first radiation is taken as quantization axis[1]). A third example is the angular distribution of radiation from an ensemble of nuclei oriented in such a way that an axis of rotational symmetry exists. The angular effects of γ-radiation in the latter case have been treated in previous chapters[2]) [3]). In all these examples one studies the angular distribution of nuclear radiation emitted by an ensemble with cylindrical symmetry, which is prepared by preceding effects.

However, the general case with non-vanishing off-diagonal density matrix elements has also physical importance for a number of phenomena. This can easily be seen in the nuclear reaction of the type (a, b, c), i.e., one incident particle a in a fixed direction, followed by two successive radiations b and c. After the absorption of particles a the nuclei form an ensemble which has no longer spherical symmetry but symmetry around the direction of a. After emission of b in a fixed direction this symmetry has also disappeared and the ensemble is described by a density matrix with non-vanishing off-diagonal elements. If one wishes to calculate the angular correlation between b and c we have the problem of finding the angular distribution of radiation c. A second example is an ensemble of nuclei which has arisen from orientation with cylindrical symmetry, and subsequent emission of a first radiation in a fixed direction. With the density matrix of this ensemble the distribution of a second radiation can be calculated (in other words the correlation of successive radiations from oriented nuclei). Finally a third example is an ensemble of nuclei which has been oriented by means of such methods, that the system has no axis of rotational symmetry [4]) [5]) [6]). The calculation of the angular distribution of radiation from such an ensemble presents again the full problem of this chapter.

For the calculation of the angular distribution of the radiation we shall use an alternative description of the ensemble from that given by the density matrix $\varrho_{mm'}$. This description is provided by Fano's statistical tensors [7]). They are defined by

$$\langle|jjkq\rangle = \Sigma_{mm'}\, \varrho_{mm'}\, (-1)^{j-m} \langle jm'j-m|\, jjkq\rangle. \qquad (3)$$

Here $\langle jm'j-m|\, jjkq\rangle$ are the transformation coefficients for the addition of angular momenta. The set of statistical tensors $\langle|jjkq\rangle$ (with $k = 0 \ldots . 2j$ and $q = -k \ldots . + k$) is equivalent to the density matrix $\varrho_{mm'}$. They are, however, more appropriate in considerations on effects of geometrical nature. In fact with the use of the statistical tensors we derive in section 2 a simple closed expression for the angular distribution of radiation from an arbitrary ensemble.

In section 3 we give formulae for a few special cases. In section 4 we indicate in more detail some possibilities for the application of the formulae obtained.

§ 2. *Derivation of the angular distribution function.* If an initial state of a nucleus is denoted by $|A\rangle$ and a final state by $|f\rangle$, the relative probability of a nuclear radiation to be emitted in the direction **k** by the ensemble is given by

$$W(\mathbf{k}) = \Sigma_f \overline{\overline{|\langle A|\, H(\mathbf{k})|\, f\rangle|^2}}. \qquad (4)$$

The double bar indicates an average over the ensemble of nuclei. The operator $H(\mathbf{k})$ is the interaction Hamiltonian for the radiation and the nucleus. Introducing in (4) the density matrix defined by (1) and (2) we obtain

$$W(\mathbf{k}) = \Sigma_{mm'm_f}\, \varrho_{mm'} \langle jm|\, H(\mathbf{k})|\, j_f m_f\rangle^* \langle jm'|\, H(\mathbf{k})|\, j_f m_f\rangle. \qquad (5)$$

We shall use the algebra of tensor operators, developed by R a c a h [8]), for the evaluation of (5). The Hamiltonian is expanded in irreducible tensor operators. In a coordinate system with **k** as z-axis, relative to which the emitted radiation is described, this expansion reads

$$\bar{H} = \Sigma_{LM}\, a_{LM}\, T^L_M. \qquad (6)$$

In (6) the a_{LM} are parameters which characterize the emitted radiation. The components T^L_M of the irreducible tensor operators of degree L operate on the nucleus. Relative to the coordinate system with the quantization axis as z axis (6) becomes

$$H(\mathbf{k}) = \Sigma_{LM\mu}\, a_{LM}\, T^L_\mu\, D^L_{\mu M}(S), \qquad (7)$$

where S is the rotation in space which transforms the coordinate system used in (6) into the coordinate system used in (7). D^L is the $(2L+1)$ dimensional irreducible representation of the rotation group. With (7) we obtain for the distribution function (5)

$$W(\mathbf{k}) =$$
$$= \Sigma\, \varrho_{mm'}\, a^*_{LM}\, a_{L'M'} \langle jm\,|T^L_\mu|\, j_f m_f\rangle^* \langle jm'\,|T^{L'}_{\mu'}|\, j_f m_f\rangle\, D^{L*}_{\mu M}(S)\, D^L_{\mu' M'}(S). \qquad (8)$$

With the definition of R a c a h's V-function

$$\langle aab\beta | \, abc\gamma \rangle = (-1)^{c+\gamma} (2c+1)^{\frac{1}{2}} \, V(abc; \alpha\beta - \gamma) \tag{9}$$

and the relations

$$D^{L*}_{\mu M}(S) \, D^{L'}_{\mu' M'}(S) =$$

$$= \Sigma_{k\varrho\sigma} (-1)^{M-\mu} \langle L - \mu L' \mu' | \, LL'k\varrho \rangle D^{k}_{\varrho\sigma}(S) \langle LL'k\sigma | \, L - ML'M' \rangle, \tag{10}$$

$$\langle jm \, | T^{L}_{\mu} | \, j_f m_f \rangle = (-1)^{j+m} \langle j \, \| T^{L} \| \, j_f \rangle \, V(jj_f L; \, -mm_f\mu), \tag{11}$$

we derive from (8)

$$W(\mathbf{k}) = \Sigma_{LL'} \langle j \, \| T^{L} \| \, j_f \rangle^{*} \langle j \, \| T^{L'} \| \, j_f \rangle \times$$

$$\times \Sigma \varrho_{mm'} a^{*}_{LM} a_{L'M'} (-1)^{m-m'+M-\mu+k+\varrho} (2k+1)^{\frac{1}{2}} \langle LL'k\sigma | L - ML'M' \rangle \times$$

$$\times D^{k}_{\varrho\sigma}(S) \, V(LL'k; -\mu\mu' - \varrho) \, V(jj_f L; -mm_f\mu) V(jj_f L'; -m'm_f\mu'). \tag{12}$$

The second summation has to be extended over m_f, m, m', M, M', μ, μ', k, ϱ and σ. Summation over μ, μ' and m_f with the aid of formula (41) from R a c a h's paper [8]) gives

$$W(\mathbf{k}) = \Sigma_{LL'} \langle j \, \| T^{L} \| \, j_f \rangle^{*} \langle j \, \| T^{L'} \| \, j_f \rangle \, \Sigma \, a^{*}_{LM} a_{L'M'} \, (-1)^{L-M+k}$$

$$\langle LL'k\sigma \, | \, L - ML'M' \rangle \times D^{k}_{\varrho\sigma}(S) \, W(j_f L' jk; \, jL) \, \Sigma_{mm'} \, \varrho_{mm'} \, (-1)^{j-m}$$

$$\langle jm'j - m | \, jjk\varrho \rangle. \tag{13}$$

The second summation is over k, ϱ, σ, M and M'. We have omitted unessential factors which only affect the normalization. The functions $W(j_f L' jk; \, jL)$ are the R a c a h coefficients [8]) [9]). With the statistical tensors $\langle | jjk\varrho \rangle$ defined by (3) and the abbreviation

$$C_{k\sigma}(LL') = \Sigma_{MM'} \, (-1)^{L-M} \, a^{*}_{LM} a_{L'M'} \, \langle LL'k\sigma | \, L - ML'M' \rangle, \tag{14}$$

the formula for the angular distribution (13) reads

$$W(\mathbf{k}) = \Sigma_{LL'} \langle j \, \| T^{L} \| \, j_f \rangle^{*} \langle j \, \| T^{L'} \| \, j_f \rangle \, \Sigma_{k\varrho\sigma} (-1)^{k} \, C_{k\sigma}(LL') \, W(j_f L' jk; jL) \times$$

$$\times \langle | jjk\varrho \rangle \, D^{k}_{\varrho\sigma}(S). \tag{15}$$

As is seen this formula contains a sum of terms of which the factors describe the relevant physical features of the problem, *viz.* two physical factors $\langle j \, \| T^{L} \| \, j_f \rangle$ and $\langle j \, \| T^{L'} \| \, j_f \rangle$ pertaining to the nuclear transition, a factor $C_{k\sigma}(LL')$ which characterizes the radiation, the Racah coefficient containing the total angular momenta, the statistical tensors which describe the initial orientation, and finally a function $D^{k}_{\varrho\sigma}(S)$ which depends only on the Euler angles of the rotation S.

§ 3. *Special cases*. For a radiation with a single value of the total angular momentum L formula (15) reads

$$W(\mathbf{k}) = \Sigma_{k\varrho\sigma} (-1)^k C_{k\sigma} (LL) W(j_f Ljk;\ jL) \langle|jjk\varrho\rangle D^k_{\varrho\sigma}(S), \quad (16)$$

where factors now irrelevant have been omitted. For the case of cylindrical symmetry of the ensemble, the density matrix has diagonal form which means that only the statistical tensors $\langle|jjk0\rangle \equiv \equiv f_k(j)$ subsist (cf. [2]) section 2). Thus a specialization of (16) is obtained:

$$W(\mathbf{k}) = \Sigma_{k\sigma} (-1)^k C_{k\sigma}(LL)\ W(j_f jLk;\ Lj)\ f_k(j)\ D^k_{0\sigma}(S), \quad (17)$$

which has been derived before [2]).

For pure 2^L-pole γ-rays, of which no polarization is observed (16) gives

$$W(\mathbf{k}_1) = \Sigma_{k\varrho} \langle L1L{-}1|\ LLk0\rangle\ W(j_f jLk;\ Lj) \langle|jjk\varrho\rangle\ D^k_{\varrho 0}(S) \quad (18)$$

with only even values of k. The value of $C_{k\sigma}(LL)$ for γ-rays has been used in (16) (cf. [2]) formula 84). Specialization to the case of cylindrical symmetry gives

$$W(\mathbf{k}) = \Sigma_k \langle L1L-1|\ LLk0\rangle\ W(j_f jLk;\ Lj)\ f_k(j)\ D^k_{00}(S) \quad (19)$$

with even values for k. $D^k_{00}(S)$ is proportional to the Legendre polynomial $P_k(\cos\vartheta)$. The result (19) has also been derived before [2]).

§ 4. *Remarks on some possible applications*. As has been indicated in section 1, the calculation of the angular correlation of two successive radiations from oriented nuclei presents a possibility for the application of formula (15). If the ensemble of nuclei has an axis $\boldsymbol{\eta}$ of rotational symmetry the density matrix is on diagonal form which corresponds to the vanishing of all quantities $\langle|jjk\varrho\rangle$ with $\varrho \neq 0$. After the emission of the first radiation in the direction $\mathbf{k}_1$ the ensemble is described by a full set of statistical tensors $\langle|jjk\varrho\rangle$ which can be calculated and depends on $\mathbf{k}_1$. This ensemble emits the second radiation in the direction $\mathbf{k}_2$. For the calculation of the angular distribution of this radiation we can make use of formula (15) by substituting the calculated values of $\langle|jjk\varrho\rangle$. The angular dependent factor referring to the direction $\mathbf{k}_2$ is $D^k_{\varrho\sigma}(T)$ where T is the rotation which transforms the coordinate system with $\mathbf{k}_2$ as z axis into that whose z axis is $\boldsymbol{\eta}$. A different procedure is obtained if we calculate the statistical tensors $\langle|jjk\varrho\rangle_{\mathbf{k}_1}$ describing the ensemble in a coordinate system whose z axis is $\mathbf{k}_1$. The relation between $\langle|jjk\varrho\rangle_{\mathbf{k}_1}$ and $\langle|jjk\varrho\rangle_\eta$ is given by

$$\langle|jjk\bar{\varrho}\rangle_{\mathbf{k}_1} = \Sigma_\varrho \langle|jjk\varrho\rangle_\eta \, D^k_{\varrho\bar{\varrho}}(S). \tag{20}$$

With the use of $\langle|jjk\varrho\rangle_{\mathbf{k}_1}$ in (15) we obtain again the angular distribution for the second radiation if we take for the angle dependent factor $D^k_{\varrho\sigma}(S^{-1}T)$. The rotation S is defined for $\mathbf{k}_1$ in the same way as T for $\mathbf{k}_2$. The formula obtained will be a natural extension of the formula for the correlation function in the case of nuclei with random orientation. In the latter case the direction $\boldsymbol{\eta}$ becomes irrelevant and only the rotation $S^{-1}T$ which transforms $\mathbf{k}_2$ in $\mathbf{k}_1$ will have physical significance.

In the field of nuclear reactions we find a possibility for application of (15) in the angular correlation between the directions of a neutron and a γ quantum emitted in the reaction $^7\mathrm{Li}(d, n)^8\,\mathrm{Be}^*(\gamma)\,^8\mathrm{Be}$ [10]. The fixed direction $\boldsymbol{\eta}$ of the incident deuterons will prepare an ensemble of (compound) nuclei with cylindrical symmetry. The angular correlation function can then be obtained along the same lines as indicated above.

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CHAPTER IV

DIRECTIONAL CORRELATION OF TWO SUCCESSIVE RADIATIONS EMITTED BY ORIENTED NUCLEI

Synopsis

The correlation probability $W(\mathbf{k}_1, \mathbf{k}_2, \boldsymbol{\eta})$ for the directions $\mathbf{k}_1$ and $\mathbf{k}_2$ of two radiations emitted in cascade by an ensemble of oriented nuclei is calculated. The unit vector $\boldsymbol{\eta}$ is an axis of rotational symmetry along which the nuclei are oriented. The result is specialized to the case of γ-γ correlations. For two dipole transitions $j \to j-1 \to j-2$ and for two quadrupole transitions $j \to j-2 \to j-4$ explicit formulae are given, which are suitable for direct evaluation.

§ 1. *Introduction.* If two radiations are emitted in cascade by an ensemble of nuclei we may ask for the probability $W(\mathbf{k}_1, \mathbf{k}_2)$ of finding the two radiations in the directions specified by the unit vectors $\mathbf{k}_1$ and $\mathbf{k}_2$. For an ensemble which has spherical symmetry it is clear that this probability can depend only on the angle ϑ determined by $\mathbf{k}_1$ and $\mathbf{k}_2$ ($\cos \vartheta = \mathbf{k}_1 . \mathbf{k}_2$). If the ensemble has only symmetry around an axis $\boldsymbol{\eta}$ the correlation function will depend on three angles determined by $\mathbf{k}_1$, $\mathbf{k}_2$ and $\boldsymbol{\eta}$. This will be the case for nuclei oriented in a certain direction. In section 2 a formula for the directional correlation $W(\mathbf{k}_1, \mathbf{k}_2, \boldsymbol{\eta})$ is derived. In the calculations we assume that there is no appreciable spin precession or reorientation after the emission of the first radiation and before the emission of the second radiation. The latter effects have been studied by Goertzel[1]) and Alder[2]).

The application of the formula obtained is limited as in general it contains unknown nuclear matrix elements. Only in pure multipole transitions the nuclear matrix elements become irrelevant. We have therefore given a specialization of our formula for the case of pure multipole γ-transitions (section 3). For dipole γ-transitions

$j \to j-1 \to j-2$ and quadrupole γ-transitions $j \to j-2 \to j-4$ formulae are given suitable for direct evaluation (section 4). In section 5 a numerical result is given for a temperature effect in the $\gamma - \gamma$ correlation of an ensemble of ^{60}Co nuclei contained in a tutton salt where two directions of orientation exist [3]).

§ 2. *The directional correlation function.* We consider an ensemble of oriented nuclei with an axis $\boldsymbol{\eta}$ of rotational symmetry. Then the orientation is described either by the relative populations a_{m_i} of the magnetic sublevels $m_i (m_i = -j_i \ldots + j_i)$ or by the statistical tensors $f_k(j_i) \equiv \langle |j_i j_i k 0 \rangle$ [4]) [5]). This ensemble emits two radiations in succession in the directions $\mathbf{k}_1$ and $\mathbf{k}_2$. For the characterization of these directions we use the rotations in space S and T which transform the coordinate systems with $\mathbf{k}_1$ and $\mathbf{k}_2$ respectively as z-axis into that whose z-axis is $\boldsymbol{\eta}$. These rotations are determined by their Euler angles. The nuclear spins of the initial, intermediate and final state are indicated as j_i, j_e and j_f respectively. The angular momentum carried off by the emitted radiation is denoted by L (in the two transitions, L_1 and L_2 respectively).

After the emission of the first radiation in a fixed direction $\mathbf{k}_1$ there is no longer cylindrical symmetry and the ensemble is described by a density matrix $\varrho_{m_e m_e'}$ or by the set of statistical tensors $\langle |j_e j_e kq \rangle$ [4]) [6]). Using a coordinate system with $\mathbf{k}_1$ as z-axis the statistical tensors for the description of the ensemble are $\langle | j_e j_e k\bar{q} \rangle_{\mathbf{k}_1}$ which are related to the former ones by

$$\langle |j_e j_e k\bar{q} \rangle_{\mathbf{k}_1} = \Sigma_q \langle |j_e j_e kq \rangle D^k_{q\bar{q}}(S). \qquad (1)$$

Here D^k is the $2k+1$ dimensional representation of the rotation group. The directional *correlation* function for $\mathbf{k}_1$ and $\mathbf{k}_2$ is now equivalent to the directional *distribution* function of the second radiation ($\mathbf{k}_2$) emitted by the ensemble characterized by $\langle |j_e j_e k\bar{q} \rangle_{\mathbf{k}_1}$ (cf. [6])). Once we have calculated these statistical tensors we find the correlation function by substitution of these tensors in formula (15) of the previous chapter. In this way we obtain for $W(\mathbf{k}_1, \mathbf{k}_2, \boldsymbol{\eta})$

$$W = S_2 \Sigma_{L_2 L_2'} \langle j_e \| T^{L_2} \| j_f \rangle^* \langle j_e \| T^{L_2'} \| j_f \rangle \times$$
$$\times \Sigma_{k\varrho\sigma} (-1)^k C_{k\sigma}(L_2 L_2') W(j_f L_2' j_e k;\, j_e L_2) \langle |j_e j_e k\varrho \rangle_{\mathbf{k}_1} D^k_{\varrho\sigma}(S^{-1}T). \qquad (2)$$

The index 2 pertains to the second radiation. S_2 indicates summation over all parameters which are not observed (e.g. polarization directions). Since $\mathbf{k}_1$ is the z-axis of the reference system, $S^{-1}T$ which transforms $\mathbf{k}_2$ into $\mathbf{k}_1$ occurs in the formula (cf. [5]) section 4).

If we begin with the general expression for the directional correlation function [7])

$$W = S_1 S_2 \, \Sigma_{m_i m_f} a_{m_i} \left| \Sigma_{m_e} \langle j_i m_i \mid H_1 \mid j_e m_e \rangle \langle j_e m_e \mid H_2 \mid j_f m_f \rangle \right|^2, \tag{3}$$

we are led to the same result as can be seen by the following transcription of (3). With

$$\varrho_{m_e m_e'} = S_1 \, \Sigma_{m_i} a_{m_i} \langle j_i m_i \mid H_1 \mid j_e m_e \rangle^* \langle j_i m_i \mid H_1 \mid j_e m_e' \rangle, \tag{4}$$

the formula (3) reads

$$W = S_2 \, \Sigma_{m_e m_e' m_f} \varrho_{m_e m_e'} \langle j_e m_e \mid H_2 \mid j_f m_f \rangle^* \langle j_e m_e' \mid H_2 \mid j_f m_f \rangle. \tag{5}$$

This expression reduces to formula (2) since it is the starting point for its derivation (cf. [6]) formula (5)).

The problem is thus restricted to the calculation of $\langle \, | j_e j_e \, k\varrho \rangle_{\mathbf{k}_1}$. The statistical tensors $\langle \, | j_e j_e \, k\bar{\varrho} \rangle$ referring to the system with $\boldsymbol{\eta}$ as z-axis are defined by

$$\langle \, | j_e j_e \, k\bar{\varrho} \rangle = \Sigma_{m_e m_e'} \varrho_{m_e m_e'} (-1)^{j_e - m_e} \langle j_e m_e' j_e - m_e \mid j_e j_e \, k\bar{\varrho} \rangle. \tag{6}$$

Here $\langle j_e m_e' j_e - m_e \mid j_e j_e \, k\bar{\varrho} \rangle$ are the transformation coefficients for the addition of angular momenta. The density matrix $\varrho_{m_e m_e'}$, is given by the expression (4). From (6) and (1) we obtain

$$\langle \, | j_e j_e \, k\varrho \rangle_{\mathbf{k}_1} = \Sigma_{m_e m_e' \bar{\varrho}} \varrho_{m_e m_e'} (-1)^{j_e - m_e} \langle j_e m_e' j_e - m_e \mid j_e j_e \, k\varrho \rangle D^k_{\bar{\varrho}\varrho}(S). \tag{7}$$

We follow in principle the same procedure as was used previously [6]). With the development of the Hamiltonian in irreducible tensor operators (cf. [6]) formulae 6—7)

$$H_1 = \Sigma_{L_1 M_1 \mu_1} a_{L_1 M_1} T^{L_1}_{\mu_1} D^{L_1}_{\mu_1 M_1}(S), \tag{8}$$

R a c a h 's definition [8]) of the functions V

$$\langle a\alpha b\beta \mid abc\gamma \rangle = (-1)^{c+\gamma} (2c+1)^{\frac{1}{2}} V(abc; \alpha\beta - \gamma), \tag{9}$$

and the relation

$$\langle j_i m_i \mid T^{L_1}_{\mu_1} \mid j_e m_e \rangle = (-1)^{j_i + m_i} \langle j_i \| T^{L_1} \| j_e \rangle V(j_i j_e L_1; -m_i m_e \mu_1), \tag{10}$$

we derive from (7) and (4)

$$\langle | j_e j_e k\varrho \rangle_{\mathbf{k}_1} = S_1 \Sigma_{L_1 L_1'} \langle j_i \| T^{L_1} \| j_e \rangle^* \langle j_i \| T^{L_1'} \| j_e \rangle \Sigma a_{m_i} a^*_{L_1 M_1} a_{L_1' M_1'} \times$$
$$\times D^{L_1 *}_{\mu_1 M_1}(S)\, D^{L_1'}_{\mu_1' M_1'}(S)\, D^k_{\bar{\varrho}\varrho}(S)\, (-1)^{j_e - m_e + k + \bar{\varrho}} (2k+1)^{\frac{1}{2}} \times$$
$$\times V(j_i j_e L_1; -m_i m_e \mu_1)\, V(j_i j_e L_1'; -m_i m_e' \mu_1')\, V(j_e j_e k; m_e' - m_e - \varrho). \quad (11)$$

The second summation in (11) must be extended over m_i, m_e, m_e', μ_1, μ_1', M_1, M_1' and $\bar{\varrho}$. Combination of the angle dependent factors in (11) according to (W i g n e r [9]))

$$D^{L_1 *}_{\mu_1 M_1}(S)\, D^k_{\bar{\varrho}\varrho}(S) =$$
$$= \Sigma_{l\lambda x} (-1)^{M_1 - \mu_1} \langle L_1 - \mu_1 k\bar{\varrho} \,|\, L_1 k l \lambda \rangle D^l_{\lambda x}(S) \langle L_1 - M_1 k\varrho | L_1 k l x \rangle, \quad (12)$$
$$D^{L_1'}_{\mu_1' M_1'}(S) D^l_{\lambda x}(S) = \Sigma_{k'\varrho'\sigma'} \langle L_1' \mu_1' l \lambda | L_1' l k' \varrho' \rangle D^{k'}_{\varrho'\sigma'}(S) \langle L_1' M_1' l x | L_1' l k' \sigma' \rangle, \quad (13)$$

makes the summation possible over m_e, μ_1 and $\bar{\varrho}$. This gives expressions containing Racah coefficients W (cf. [8]) formula 41). Applying again Racah's formula for the summation of m_e', μ_1' and λ, the result is

$$\langle | j_e j_e k\varrho \rangle_{\mathbf{k}_1} = S_1 \Sigma_{L_1 L_1'} \langle j_i \| T^{L_1} \| j_e \rangle^* \langle j_i \| T^{L_1'} \| j_e \rangle \times$$
$$\Sigma_{M_1 M_1' \sigma' k' l x} (-1)^{L_1 - M_1} a^*_{L_1 M_1} a_{L_1' M_1'} \langle L_1 - M_1\, k\varrho | L_1\, k l x \rangle \cdot$$
$$\cdot \langle L_1' M_1' \, l x \,|\, L_1' l k' \sigma' \rangle \times \quad (14)$$
$$(2l+1)^{\frac{1}{2}} (2k+1)^{\frac{1}{2}}\, W(j_e j_i\, l k'; L_1' j_i)\, W(j_e j_i\, k l; L_1 j_e) \langle | j_i j_i\, k' 0 \rangle D^{k'}_{0\sigma'}(S).$$

The formulae (2) and (14) provide the directional correlation function for two arbitrary radiations emitted in succession by an ensemble of oriented nuclei. The resulting correlation function is not normalized since normalisation factors have not been taken into account in (4).

Remark. From (2) and (14) a more symmetrical form can be obtained for the correlation function $W(\mathbf{k}_1, \mathbf{k}_2, \boldsymbol{\eta})$ which reads

$$W(\mathbf{k}_1, \mathbf{k}_2, \boldsymbol{\eta}) = S_1 S_2 \Sigma_{L_1 L_1' L_2 L_2'} \langle j_i \| T^{L_1} \| j_e \rangle^* \langle j_i \| T^{L_1'} \| j_e \rangle \langle j_e \| T^{L_2} \| j_f \rangle$$
$$\langle j_e \| T^{L_2'} \| jf \rangle \Sigma_{kk'k''l\sigma\sigma''} (-1)^{k+k'} C_{k''\sigma''}(L_1 L_1') C_{k\sigma}(L_2 L_2') (2k''+1)^{1/2} (2k+1)^{1/2}$$
$$\langle | j_i j_i\, k' o \rangle (2l+1)\, W(j_e j_i\, kl;\, L_1 j_e)\, W(j_e j_i\, L_1' k';\, l j_i)\, W(l L_1'\, k k'';\, k' L_1)\, .$$
$$.\, W(j_f L_2' j_e\, k;\, j_e L_2)\, \Sigma_\varrho \langle k'' - \varrho k\varrho | \, k'' k k' 0 \rangle D^{k''}_{-\varrho\sigma''}(S)\, D^k_{\varrho\sigma}(T).$$

§ 3. *Specialization to gamma radiation*. In the case of pure multipole γ-transitions we can use the explicit expressions for a_{LM} ([5])

section 5) and omit the nuclear matrix elements, which are now irrelevant. In this way we obtain from (14)

$$\langle |j_e j_e k\varrho\rangle_{\mathbf{k}_1} = \Sigma_{M_1=\pm 1, k' l x} (-1)^{L_1-M_1} \langle L_1 - M_1 k\varrho | L_1 k l x\rangle \langle L_1 M_1 l x | L_1 l k' \varrho\rangle$$
$$(2l+1)^{\frac{1}{2}} (2k+1)^{\frac{1}{2}} W(j_e j_i l k'; L_1 j_i)\, W(j_e j_i k l; L_1 j_e) \langle |j_i j_i\, k' 0\rangle D^{k'}_{0\varrho}(S). \quad (15)$$

To obtain the correlation function we must substitute (15) in the formula (2) specialized to the case under consideration (cf. [6]) formula (18)) i.e.

$$W = \Sigma_{k\varrho} \langle L_2 1 L_2 - 1 | L_2 L_2\, k0\rangle\, W(j_f j_e L_2 k; L_2 j_e) \langle |j_e j_e k\varrho\rangle_{\mathbf{k}_1} D^k_{\varrho 0}(S^{-1}T). \quad (16)$$

If we use the notation (ref. [5]) formulae 23, 25 and 26)

$$f_{k'}(j_i) = w_{k'}(j_i) \langle |j_i j_i\, k'0\rangle \quad (17)$$

in (15) and insert it in (16) we obtain the correlation function in the following form [10])

$$W(\mathbf{k}_1, \mathbf{k}_2, \boldsymbol{\eta}) = \Sigma_{k'} f_{k'}(j_i)\, \Sigma_{k\varrho} C^{k'k}_{\varrho}(j_i j_e j_f L_1 L_2)\, D^{k'}_{0\varrho}(S)\, D^k_{\varrho 0}(S^{-1}T). \quad (18)$$

In (18) we have combined a number of factors to one function $C^{k'k}_{\varrho}(j_i j_e j_f L_1 L_2)$.

Remark. A specialization to *randomly* oriented nuclei must give the ordinary correlation function. In fact one obtains in this case from (15) for k is even

$$\langle |j_e j_e\, k0\rangle_{\mathbf{k}_1} = \langle L_1\, 1 L_1 - 1 | L_1 L_1\, k0\rangle\, W(j_e j_i k L_1; L_1 j_e) \quad (19)$$

which has been indicated before ([11]) formula 30).

§ 4. *Explicit expressions for the correlation function for gamma radiation in two special cases.* For the use of numerical evaluation we have applied formula (15) and (16) for γ radiation to the case of two successive dipole transitions $j_i \to j_i - 1 \to j_i - 2$. The angle dependent functions are expressed in the scalar products of the three characteristic unit vectors $\boldsymbol{\eta}$, $\mathbf{k}_1$, $\mathbf{k}_2$. The result is [12])

$$W(\mathbf{k}_1, \mathbf{k}_2, \boldsymbol{\eta}) = 9\,[N_0(j_i)\, f_0(j_i)\, F_0(\mathbf{k}_1\mathbf{k}_2) + N_2(j_i) f_2(j_i)\, F_2(\mathbf{k}_1\mathbf{k}_2\boldsymbol{\eta}) + $$
$$+ N_4(j_i)\, f_4(j_i)\, F_4(\mathbf{k}_1\mathbf{k}_2\boldsymbol{\eta})]. \quad (20)$$

The following abbreviations are used:

$$F_0(\mathbf{k}_1\mathbf{k}_2) \equiv \tfrac{1}{30}\,[13 + \alpha^2], \quad (21a)$$

$$F_2(\mathbf{k}_1\mathbf{k}_2\boldsymbol{\eta}) \equiv -\tfrac{1}{7}[4 - 6\alpha_1^2 - 6\alpha_2^2 - 3\alpha\alpha_1\alpha_2 + \alpha^2], \quad (21b)$$

$$F_4(\mathbf{k}_1\mathbf{k}_2\boldsymbol{\eta}) \equiv \tfrac{1}{8}\,[1 - 5a_1^2 - 5a_2^2 + 35a_1^2 a_2^2 - 20aa_1a_2 + 2a^2], \quad (21c)$$

$$a \equiv (\mathbf{k}_1.\mathbf{k}_2), \quad a_1 \equiv (\mathbf{k}_1.\boldsymbol{\eta}), \quad a_2 \equiv (\mathbf{k}_2.\boldsymbol{\eta}), \quad (22)$$

$$N_k(j) \equiv 2^{\frac{1}{2}k}\, j^k (2j - k)!/(2j)! \qquad (k \text{ is even}). \quad (23)$$

Explicit expressions for the orientation parameters $f_k(j)$ for $k = 0$, 1, 2, 3, 4 have been given before (ref. [5]) section 2). The function $W(\mathbf{k}_1, \mathbf{k}_2, \boldsymbol{\eta})$ given by (20) is normalized to

$$\int W(\mathbf{k}_1, \mathbf{k}_2, \boldsymbol{\eta})\, \mathrm{d}\Omega_1\, \mathrm{d}\Omega_2 = (8\pi)^2. \quad (24)$$

We have also applied formulae (15) and (16) to the case of two quadrupole transitions $j_i \to j_i - 2 \to j_i - 4$. The result is [12])

$$W(\mathbf{k}_1, \mathbf{k}_2, \boldsymbol{\eta}) = 25\,[N_0(j_i)\, f_0(j_i)\, G_0(\mathbf{k}_1\mathbf{k}_2) + N_2(j_i)\, f_2(j_i)\, G_2(\mathbf{k}_1\,\mathbf{k}_2\,\boldsymbol{\eta}) + \\ + N_4(j_i)\, f_4(j_i)\, G_4(\mathbf{k}_1\,\mathbf{k}_2\,\boldsymbol{\eta}) + N_6(j_i)\, f_6(j_i)\, G_6(\mathbf{k}_1\mathbf{k}_2\boldsymbol{\eta}) + \\ + N_8(j_i)\, f_8(j_i)\, G_8(\mathbf{k}_1\mathbf{k}_2\boldsymbol{\eta})]. \quad (25)$$

The functions G_k are defined by

$$G_0(\mathbf{k}_1\mathbf{k}_2) \equiv \tfrac{1}{315}\,[48 + 6a^2 + 2a^4], \quad (26a)$$

$$G_2(\mathbf{k}_1\mathbf{k}_2\boldsymbol{\eta}) \equiv \tfrac{1}{231}\,[96 - 144a_1^2 - 144a_2^2 + 36aa_1a_2 + 18a^2a_1^2 + \\ + 18a^2a_2^2 - 24a^2 + 24a^3a_1a_2 - 8a^4], \quad (26b)$$

$$G_4(\mathbf{k}_1\mathbf{k}_2\boldsymbol{\eta}) \equiv \tfrac{1}{286}\,[-201 + 1005a_1^2 + 1005a_2^2 - 1225a_1^4 - 1225a_2^4 + \\ + 315a_1^2a_2^2 - 540aa_1a_2 + 420aa_1^3a_2 + 420aa_1a_2^3 + \\ + 108a^2 - 270a^2a_1^2 - 270a^2a_2^2 + 630a^2a_1^2a_2^2 - 360a^3a_1a_2 + 36a^4], \quad (26c)$$

$$G_6(\mathbf{k}_1\mathbf{k}_2\boldsymbol{\eta}) \equiv \tfrac{1}{60}\,[-6 + 63a_1^2 + 63a_2^2 - 756a_1^2a_2^2 - 63a_1^4 - 63a_2^4 + \\ + 693a_1^4a_2^2 + 693a_1^2a_2^4 + 504aa_1a_2 - 1008aa_1^3a_2 - \\ - 1008aa_1a_2^3 + 1848aa_1^3a_2^3 - 48a^2 + 252a^2a_1^2 + \\ + 252a^2a_2^2 - 1512a^2a_1^2a_2^2 + 336a^3a_1a_2 - 16a^4], \quad (26d)$$

$$G_8(\mathbf{k}_1\mathbf{k}_2\boldsymbol{\eta}) \equiv \tfrac{1}{32}[3 - 54a_1^2 - 54a_2^2 + 1188a_1^2a_2^2 + 99a_1^4 + 99a_2^4 - 2574a_1^4a_2^2 - \\ - 2574a_1^2a_2^4 + 6435a_1^4a_2^4 - 432aa_1a_2 + 1584aa_1^3a_2 + \\ + 1584aa_1a_2^3 - 6864aa_1^3a_2^3 + 24a^2 - 216a^2a_1^2 - \\ - 216a^2a_2^2 + 2376a^2a_1^2a_2^2 - 288a^3a_1a_2 + 8a^4]. \quad (26e)$$

The a, a_1 and a_2 are again defined by (22). The explicit expressions for the orientation parameters $f_k(j)$ for $k = 6{,}8$ are

$$f_6 = (1/j^6)\,[\Sigma_m m^6 a_m - (5/11)\,(3j^2 + 3j - 7)\,\Sigma_m m^4 a_m + \\ + (1/11)\,(5j^4 + 10j^3 - 20j^2 - 25j + 14)\,\Sigma_m m^2 a_m - \\ - (5/231)\,j(j+1)\,(j-1)\,(j+2)\,(j-2)\,(j+3)]. \quad (27a)$$

$$f_8 = (1/j^8)\,[\Sigma_m m^8 a_m - (14/15)\,(2j^2+2j-9)\,\Sigma_m m^6 a_m + (7/39)\,(6j^4+12j^3 - \\ -50j^2 - 56j + 81)\,\Sigma_m m^4 a_m - (7/2145)\,(60j^6 + 180j^5 - 690j^4 - 1680j^3 + \\ + 1958j^2 + 2828j - 9132/7)\,\Sigma_m m^2 a_m + \\ + (7/1287)\,j(j+1)\,(j-1)\,(j+2)\,(j-2)\,(j+3)\,(j-3)\,(j+4)]. \quad (27b)$$

The correlation function (25) is normalized in the same way (24) as in the dipole case.

For randomly oriented nuclei ($f_k = 0$ for $k = 1, 2, \ldots$) we obtain from (20) and (25) the usual correlation functions.

For total orientation both (20) and (25) can be written as

$$W(\mathbf{k}_1, \mathbf{k}_2, \boldsymbol{\eta}) = W(\mathbf{k}_1, \boldsymbol{\eta})\,.\,W(\mathbf{k}_2, \boldsymbol{\eta}). \quad (28)$$

Here $W(\mathbf{k}_1, \boldsymbol{\eta})$ and $W(\mathbf{k}_2, \boldsymbol{\eta})$ are the angular distributions of the radiations separately. Hence for total orientation there is no directional correlation in these cases. From the general formula it follows that no correlation will be left for total orientation if we have a 2^{L_1}-pole and 2^{L_2}-pole radiation in succession and if

$$j_i \rightarrow j_i - L_1 \rightarrow j_i - L_1 - L_2.$$

§ 5. *A temperature effect in the γ—γ correlation for ^{60}Co.* Oriented ^{60}Co nuclei present a possibility for the application of formula (25) since they emit after a β-transition ($j_0 = 5 \rightarrow j_i = 4$) two successive γ quanta of quadrupole character in the transitions $j_i \rightarrow j_i - 2 \rightarrow \rightarrow j_i - 4$. As was indicated before (cf. [13]) formula 15 and section 3) the quantities $N_k(j_i)\, f_k(j_i)$, occurring in (25) are related to the corresponding ones before the β-transition by the relation.

$$N_k(j_i)\, f_k(j_i) = N_k\,(j_0)\, f_k(j_0). \quad (29)$$

For the initial population of the sublevels (j_0, m_0) we assume to have

$$a_{m_0} = A \cosh \beta\, m_0, \quad (30)$$

where A is determined by

$$\Sigma_{m_0=-j_0}^{+j_0}\, a_{m_0} = 1, \quad (31)$$

and the parameter β by $\beta = \mu B/kTj_0$ (cf. [5]) formula 20). The cobalt is contained in a tutton salt and due to the crystalline field the orientation is such that the levels — m_0 and m_0 are equally favoured (this is also expressed by formula (30)). As a consequence only $f_k(j_0)$ with k even subsist (*alignment* [11])). The values of $f_k(j_0)$ have been calculated as a function of β. With (29) the $N_k(j_i)\, f_k(j_i)$ are derived from the $f_k(j_0)$. For any direction of $\mathbf{k}_1$ and $\mathbf{k}_2$ the correlation function (25) can now be evaluated as a function of β. However a complication arises from the existence of two directions of orientation in the crystal. We specify them by $\boldsymbol{\eta}_1$ and $\boldsymbol{\eta}_2$ with $(\boldsymbol{\eta}_1 \cdot \boldsymbol{\eta}_2) = \cos 68°$ [3]). Along each axis half of the nuclei are oriented and we have added the corresponding correlation functions of the two ensembles of nuclei in order to obtain the observed correlation. The direction $\mathbf{k}_1$

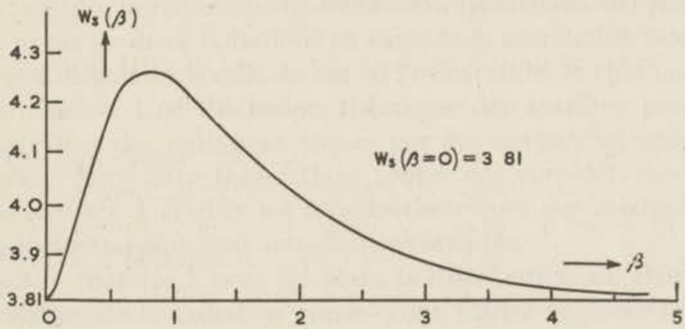

Fig. 1. The correlation function $W_s(\beta)$ plotted as a function of β.

is chosen perpendicular to the plane of $\boldsymbol{\eta}_1$ and $\boldsymbol{\eta}_2$ and $\mathbf{k}_2$ in the plane of $\boldsymbol{\eta}_1$ and $\boldsymbol{\eta}_2$ with $(\mathbf{k}_1 \cdot \boldsymbol{\eta}_1) = \cos 20°$ and $(\mathbf{k}_2 \cdot \boldsymbol{\eta}_2) = \cos 88°$. The correlation function $W_s\,(\beta)$ for this choice of angles is only a function of β. The result is shown in fig. 1. Due to the particular choice of $\mathbf{k}_2$ one has $W_s(\beta = 0) = W_s(\beta = \infty)$; this means that the correlation probabilities for randomly oriented nuclei ($\beta = 0$) and for totally oriented nuclei ($\beta = \infty$) are equal. As can be seen from the figure, $W_s(\beta)$ has a maximum of about 12 % for $\beta = 0.8$. Thus, if a sufficiently high degree of orientation could be obtained, the correlation probability would first increase and then decrease as the apparatus warmed up during an experiment.

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SOMMAIRE

Lorsqu'on considère un ensemble de noyaux radioactifs dont la distribution dans l'espace des directions du spin n'est pas à symétrie sphérique (orientation nucléaire), on peut s'attendre à des effets angulaires quant au rayonnement émis par les noyaux. En premier lieu la distribution angulaire de la radiation émise ne sera pas isotrope; ensuite la radiation elle-même sera (partiellement) polarisée. Dans le cas de deux radiations en cascade la corrélation directionnelle sera différente de celle du cas où l'orientation du spin nucléaire est arbitraire. Une discussion théorique des susdites propriétés géométriques des radiations émises par des noyaux orientés a été présentée dans cette thèse. Dans toutes nos considérations nous avons renoncé à étudier les aspects théoriques des methodes par lesquelles les noyaux sont actuellement orientés.

Dans le chapitre I nous calculons la distribution angulaire et la polarisation de la radiation émise. Pour traiter ce problème nous avons supposé que l'orientation nucléaire a un axe de symétrie rotationnelle. Il s'est avéré que le choix des paramètres pour la description de l'orientation nucléaire et pour la polarisation du rayonnement a une grande importance si l'on veut aboutir a des formules qui mettent en évidence les différents aspects physiques du problème. Pour le cas d'une radiation γ dipolaire ou quadrupolaire des formules sont présentées qui permettent un calcul numérique par des substitutions simples.

Dans le chapitre II nous avons étudié le cas où la radiation γ observée est précédée d'une transition β ou γ. L'influence d'une pareille transition sur la distribution angulaire et la polarisation de la radiation γ est indiquée. Comme cette situation se présente fréquemment dans la pratique, les effets calculés auront une certaine importance. A titre d'exemple le procédé du calcul de la distribution angulaire du rayonnement γ, tout en tenant compte de la transition β précédante, est indiqué pour des noyaux ^{60}Co orientés. Dans le

même chapitre nous avons indiqué quelles grandeurs physiques peuvent être obtenues (au moins en principe) par la mesure du rayonnement émis par des noyaux orientés.

Dans le chapitre III nous nous sommes libérés de la supposition que l'ensemble des noyaux est à symétrie rotationelle (chapitre I). La distribution angulaire est calculée pour le cas général d'un rayonnement émis par un ensemble arbitraire.

Utilisant les résultats obtenus du chapitre III nous étudions dans le chapitre IV la corrélation directionnelle de deux radiations successives émises par un ensemble de noyaux orientés en supposant de nouveau l'existence d'un axe de symétrie rotationnelle. Dans un cas particulier de la radiation γ dipolaire ou quadrupolaire nous avons donné des formules explicites qui se prêtent à des calculs numériques.

SAMENVATTING

Indien men een verzameling van radioactieve kernen beschouwt, waarbij de verdeling der spinrichtingen in de ruimte niet isotroop is (gerichte kernen), kan men hoekeffecten verwachten wat betreft de uitgezonden straling. In de eerste plaats zal de intensiteit van de kernstraling niet in alle richtingen dezelfde zijn. In de tweede plaats zal de straling zelf (gedeeltelijk) gepolariseerd zijn. Voor het geval er twee stralingen in cascade worden uitgezonden zal bovendien de richtingscorrelatie tussen deze twee afwijken van de correlatie die geldt voor niet gerichte kernen.

In dit proefschrift wordt een theoretische behandeling gegeven van bovengenoemde geometrische eigenschappen van de straling (speciaal γ-straling) van gerichte kernen. De theoretische aspecten van de experimentele methoden om kernen te richten worden buiten beschouwing gelaten.

In hoofdstuk I wordt de hoekverdeling en polarisatie berekend van de uitgezonden kernstraling. In dit hoofdstuk wordt verondersteld, dat de kernen zodanig zijn geörienteerd, dat er een as van rotatiesymmetrie bestaat. Het blijkt, dat de keuze der parameters voor de beschrijving van de orientatie der kernen en de polarisatie der straling van groot belang is voor het verkrijgen van formules, die overzichtelijk de verschillende physische kanten van het probleem weergeven. Hieraan zijn twee afzonderlijke paragraphen gewijd. Voor γ-dipool- en quadrupoolstraling worden formules gegeven, die geschikt zijn voor numerieke berekeningen.

In hoofdstuk II wordt het geval bestudeerd waarbij een β- of γ-overgang voorafgaat aan de waargenomen γ-straling, hetgeen practisch altijd de werkelijke situatie zal zijn. Als voorbeeld wordt voor gerichte ^{60}Co kernen (β-overgang gevolgd door twee γ-overgangen) aangegeven hoe de berekening van de hoekverdeling der γ-straling kan geschieden. Aan het einde van dit hoofdstuk worden een aantal belangrijke physische grootheden genoemd, die in principe uit de meting van de straling van gerichte kernen te verkrijgen zijn.

In hoofdstuk III wordt de veronderstelling, dat er een as van rotatiesymmetrie bestaat (hoofdstuk I) opgegeven en voor een willekeurige verzameling kernen de hoekverdeling van de uitgezonden straling berekend.

Met behulp van de resultaten van hoofdstuk III wordt in hoofdstuk IV de richtingscorrelatie berekend van twee successieve stralingen, uitgezonden door geörienteerde kernen, waarbij weer cylindersymmetrie verondersteld wordt. Voor het geval van γ-dipool- en quadrupoolstraling worden formules gegeven, die geschikt zijn voor numerieke berekeningen.

STELLINGEN

I

De numerieke berekening van de hoekverdeling van de γ -straling, uitgezonden door gerichte ^{58}Co-kernen, laat zien dat het ook experimenteel tot de mogelijkheden moet worden gerekend om hieruit inlichtingen te verkrijgen omtrent het karakter van de β -overgang.

J. A. M. Cox, S. R. de Groot and Chr. D. Hartogh, Physica **19** (1953) 1119.

II

De door Thirion gevolgde methode van berekening van de (n, γ) of (p, γ) hoekcorrelatie bij $(d, n\ \gamma)$ respectievelijk $(d, p\ \gamma)$ kernreacties bevat een principiële fout. Hierdoor is de door Thirion gegeven interpretatie van zijn experimenten alleen juist als de deutonen zonder baanimpulsmoment worden ingevangen.

J. Thirion, Thèse, Strasbourg, 1951, Ann. Phys. **8** (1953) 489.

III

Bij de interpretatie van de hoekverdeling van neutronen of protonen in (d, p) respectievelijk (d, n) kernreacties, waarbij zowel een strippingproces mogelijk is als de vorming van een tussenkern, is er geen reden om interferentie tussen deze twee processen uit te sluiten.

C. D. Swartz and J. S. Pruitt, rep. Johns Hopkins University AEC Contract No. AT (30–1) 825, 1953.

IV

De door Van der Merwe berekende kernstralen zijn ongeveer tweemaal zo groot als de waarden, die uit andere beschouwingen worden gevonden. De experimentele gegevens zijn dientengevolge aangepast met waarden van de parameters, die physisch weinig bevredigend zijn.

J. H. van der Merwe, Thesis, Leiden, 1952.

V

De beschouwingen van Poppema geven de situatie, wat betreft de circulaire polarisatie van de γ -straling, uitgezonden door gerichte kernen en het meten hiervan, niet geheel juist weer.

O. J. Poppema, Thesis, Groningen, 1953.

VI

De door Van Dranen geponeerde hypothese over het kritisch punt is formeel en physisch bezien niet ondubbelzinnig.

J. van Dranen, J. Chem. Phys. **20** (1952) 1175; **21** (1953) 567.

VII

De methode van Beattie en medewerkers om de Lennard Jones-parameters van het intermoleculaire potentiaalveld te berekenen uit de tweede viriaalcoëfficiënt is omslachtiger en geeft minder nauwkeurige resultaten dan de graphische methode.

James A. Beattie, Roland J. Barriault and James S. Brierley, J. Chem. Phys. **19** (1951) 1222; **20** (1952) 1615.

VIII

De redenering van Enskog, waaruit volgt dat de formules voor de warmtegeleiding en viscositeit van verdichte gassen ook nog gelden indien de moleculen harde bollen met een attractieveld zijn, is onjuist. Of een vergelijking van zijn theorie met de experimenten zinvol is, zal dan ook niet alleen van het meetgebied van de dichtheid, maar ook sterk van het meetgebied van de temperatuur afhangen.

D. Enskog, Kungl. Svensk. Vetensk. Akad. Handl. **63** (1912) no. 4.

IX

De door Gibert gegeven „zuiver thermodynamische" afleiding van de Onsager-relaties voor irreversibele processen is onjuist.

R. Gibert, Comptes Rendus, Paris, **256** (1953) 2145.

X

Whalley en medewerkers bepalen uit hun eigen meetresultaten en die van andere auteurs de gemiddelde waarden van de viriaalcoëfficiënten voor argon. De wijze waarop zij de gemiddelde waarden berekenen berust ten dele op onjuiste gronden.

E. Whalley, Y. Lupien and W. G. Schneider, Can. J. Chem. **31** (1953) 722.

VI

De door Van Deemter opgestelde hypothese voor het kritisch punt (gepostuleerd) en dynamisch hebben uit verdeling.

J. van Deemter J. Chem. Phys. 20 (1952) 1130; 21 (1953) 362.

VII

De methode van Beattie en medewerkers om de Lennard Jones-parameters van het intermoleculaire potentiaalveld te berekenen uit de tweede viriaalcoëfficiënt is onnauwkeuriger en geeft minder nauwkeurige resultaten dan de grafische methode.

James A. Beattie, Roland J. Barriault and James S. Brierley, J. Chem. Phys. 19 (1951) 1222; 20 (1952) 1615.

VIII

De behandeling van Enskog, waaruit volgt dat de formules voor de warmtegeleiding en viscositeit van verdichte gassen ook nog gelden indien de moleculen harde bollen met een attractieveld zijn, is onjuist. Of een vergelijking van zijn theorie met de experimenten zinvol is, zal dan ook niet alleen van het nauwkeurigheid van de dichtheid, maar ook sterk van het meetgebied van de temperatuur afhangen.

D. Enskog, Kungl. Svenska Vetensk. Akad. Handl. 63 (1922) no. 4.

IX

De door Gilbert gegeven „zuiver thermodynamische" afleiding van de Onsager-relaties voor irreversibele processen is onjuist.

M. Gilbert, Comptes Rendus, Paris. 236 (1953) 2145.

X

Whalley en medewerkers beginnen als hun eigen meetresultaten en die van anderen alleen de gemiddelde waarden van de viriaalcoëfficiënten voorop. De wijze waarop zij de gemiddelde waarden berekenen berust ten dele op onjuiste gronden.

E. Whalley, Y. Lupien and W. G. Schneider, Can. J. Chem. 31 (1953) 722.