Finite-Size Results for SU(3) Gauge Theory

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(Received 15 September 1986)

We give the semiclassical evaluation of the energy of electric flux in units of the 0^{++} glueball mass for SU(3) gauge theory and predict a crossover in this quantity around $z = 1.6$. The parameter z is the length of the sides of a cubic volume (the three-torus) in units of the inverse glueball mass. We discuss lattice gauge theory in the context of a finite volume and compare our analytic prediction with recent SU(3) Monte Carlo results.

PACS numbers: 11.15.Ha, 12.38.Gc

Recently much progress has been made in understanding finite-size effects in SU(N) gauge theories by use of analytic techniques. These analytic results are all based on the Hamiltonian picture, with gauge fields defined on a three-torus, which for convenience only is chosen to be a *cube* with sides L. The gauge fields satisfy periodic boundary conditions. This therefore describes a system with zero magnetic flux[1] and is the relevant framework for comparison with Monte Carlo calculations on a finite lattice.[2] In these Monte Carlo calculations, one likes to be close to the continuum by choosing a small lattice spacing a (through the renormalization group by choosing a small coupling constant g_0). But since computer limitations force us to restrict ourselves to a finite number of lattice sites (say N_s in each spatial direction and N_t in the time direction), the necessity of small a competes with requiring a large value for $L = N_s a$, the size of the volume.

Essential progress was made when it was realized that there could be a window in a (or g_0) for which both lattice artifacts and finite size effects are small.[3] Further progress was hampered by the lack of a *precise* understanding of both types of corrections. Despite a tremendous increase in computer power over the last decade, closer scrutiny and recently developed awareness based on the nonperturbative β function and the lattice artifacts in large scale computations[4,5] make it difficult to keep the window open. However, the form of the nonperturbative β function is irrelevant for *dimensionless* ratios. This was dramatically clear in the reanalysis[6] of data[7] for the "string tension" K_L (for a further discussion on the meaning of K_L, see below) and the mass gap M_L in units of $\Lambda_{\rm latt}$ as a function of the universal scale parameter[8] $z = M_L L$. If one considers the ratio $\sqrt{K_L}/M_L$ first introduced in Ref. 6, one obtains close to universal behavior, even at points for which $\sqrt{K_L}/\Lambda_{\rm latt}$ or $M_L/\Lambda_{\rm latt}$ separately would not support universal behavior, if the two-loop β function were used to convert lattice to physical units. In conclusion this implies that the bulk of the deviation of the data for $\sqrt{K_L}/\Lambda_{\rm latt}$ and $M_L/\Lambda_{\rm latt}$ from a universal curve is due to a substantial deviation of the two-loop β function from its nonperturbative counterpart, whereas lattice artifacts seem to be absent within the quoted errors.

Consequently, it is easier to make a connection between Monte Carlo lattice results and continuum gauge theories in a finite volume. Since in a sufficiently small volume the effective renormalized coupling constant is small, analytic calculations are feasible and one can attempt to make a quantitative comparison. This could provide a desirable confidence in Monte Carlo calculations and can be used to work oneself outward *reliably* to larger distances. At the same time these reliable Monte Carlo results as a function of the universal scale parameter z will provide valuable information on the physics of non-Abelian gauge theories. One important issue in this approach would be to decide from what value of z onwards the large-volume expansion for the mass gap[9,10] is valid. It would allow total control over the infinite-volume glueball mass determination.

Less ambitiously, we will settle for a good control over the small-volume behavior. Perturbation theory was first set up by Lüscher.[11] The theory is complicated by the (self-) interacting zero-momentum modes. The low-lying energy levels (E) can be calculated as a power series in $g^{2/3}$, where g is the renormalized coupling constant for SU(N) gauge theories at a scale L [in the modified minimal-subtraction ($\overline{\rm MS}$) scheme]:

$$g^{-2}(N,L) = -11N\ln(\Lambda L)/(24\pi^2) + \cdots, \tag{1}$$

$$E(N,L) = L^{-1}\sum_{k=1}^{\infty} \varepsilon_k(N) g(N,L)^{2k/3}. \tag{2}$$

For SU(2) the numerical values of ε_k for $k = 1-4$ and for a few low-lying energy levels were calculated by Lüscher and Münster[12] and recently for SU(3) by Weisz and Ziemann.[13] These results are collected in Table I for the ground state and the first excited 0^{++} state. The difference between the ground-state energy and the first excited 0^{++} energy is the 0^{++} glueball mass $M_L(0^{++})$, and thus gives the universal expansion parameter $z = M_L(0^{++})L$ as a power series in $g^{2/3}$. These results are valid as long as tunneling between different vacua is suppressed. Gauge transformations, periodic up to an element of the center of the gauge group, map one vacuum into the other. These are the gauge transformations considered by 't Hooft in defining electric flux on the torus.[1] The energy splitting ΔE of the ground state can therefore be identified as the energy of 't Hooft-type electric flux. Once the energy of electric flux becomes appreciable the perturbative results for the energy levels break down.

TABLE I. Coefficients for the ground state and the first excited 0^{++} state, defined in Eq. (3) for SU(2) (Ref. 12) and SU(3) (Ref. 13).

	$\varepsilon_1(N)$	$\varepsilon_2(N)$	$\varepsilon_3(N)$	$\varepsilon_4(N)$
SU(2)	4.116719735	−1.174516027	−0.118933	−0.03148
	6.3863588	−1.9720438	−0.438	−0.176
SU(3)	12.5887	−4.0628	−0.4280	−0.104
	15.38	−4.79	−0.57	−0.08

This energy of electric flux was considered elaborately in previous work[6,14,15] for SU(2), whereas in Ref. 15 also the leading result for arbitrary SU(N) was given. In this Letter we give the result (details will be published separately) beyond this leading order:

$$\Delta E(L) = 2L^{-1}\sin^2\left(\frac{\pi}{N}\right)|B_N|^2\lambda_N g(N,L)^{5/3}\exp\{-S_N g(N,L)^{-1} + T_N[\varepsilon_1(N)-\varepsilon_1(N-1)]g(N,L)^{-1/3}\}, \tag{3}$$

up to a relative error which is a positive power of the renormalized coupling constant. In this formula we have the following expressions for S_N (the tunneling action), T_N (the tunneling time), and λ_N (a contribution from transverse fluctuations[15,16])

$$S_N = \frac{2(N-1)}{2N^{1/2}}S, \quad S = 12.4637\ldots, \tag{4a}$$

$$T_N \frac{2}{2N^{1/2}}T, \quad T = 3.9186\ldots, \tag{4b}$$

$$\lambda_N = \frac{2N^{1/2}}{2(N-1)}\lambda, \quad \lambda = 0.6997\ldots. \tag{4c}$$

Unfortunately B_N and $\varepsilon_1(N)$ have no simple N dependence. By definition $\varepsilon_1(1)=0$. In Ref. 6 it was found that $B_2 = 0.206$; B_3 is expected to be of the same order of magnitude and can in principle be calculated from the wave function for the ground state found in Ref. 13. By use of Eq. (2) and Table I, Eq. (3) can be converted into an equation for $\mathcal{E}(z) = \Delta E(L)/M_L(0^{++})$.

We will now discuss the consequences of Eq. (3). Since the two terms in the exponent of Eq. (3) differ in sign one obtains as a good approximation for the point of crossover beyond which ΔE becomes appreciably different from zero

$$g_c^2(N) = \left[\frac{(N-1)S}{[\varepsilon_1(N)-\varepsilon_1(N-1)]T}\right]^3. \tag{5}$$

Using of the values of Eq. (4a), (4b), and Table I we obtain $g_c(2)^2 = 0.461$ and $g_c(3)^2 = 0.423$. The tunneling is, unlike for the ordinary instantons, through a *quantum-induced* potential barrier.[6,14] This is why the tunneling sets in at *small* values of the renormalized coupling constant. Using of the expansion of z in powers of $g^{2/3}$ which can be obtained from Table I, we predict the crossover for SU(3) to occur at $z \approx 1.6$ [for SU(2) we verify $z \approx 1.1$[6]]. It should be pointed out that the glueball mass which enters the definition of z, is also affected by tunneling. Our prediction does not take this into account; neither do we take into account higher-order corrections to Eq. (3) which might be larger than we hope for the values of g^2 we consider.

We will now compare with existing Monte Carlo data. It is essential that the lattice is sufficiently big in the time direction, since the analytic results are in the Hamiltonian formulation at zero temperature. How large the lattice has to be in the time direction is determined by the energy resolution one wants to achieve. If tunneling has set in, all masses are of the order of the glueball mass. In that region a safe criterion is to have $N_t z/N_s \gg 1$ (i.e., the temperature in units of the 0^{++} glueball mass is small). This criterion is satisfied for the calculations of Berg, Billoire, and Vohwinkel[17] and Patel *et al.*[18] If the condition is not satisfied (i.e., $z \ll N_s/N_t$) the Monte Carlo data can presumably be compared instead with the results of Coste *et al.*[19] However, it is critical to notice, if one is interested in the energy of electric flux, that when tunneling is suppressed the energy resolution must be of the order of ΔE, which is very small. This is a practical obstacle to performing Monte Carlo calculations of ΔE at values of z smaller than the value where tunneling sets in. It was realized in Ref. 6 (see also Ref. 15) that Monte Carlo data for time-time correlation functions of the spatial Polyakov loops in the fundamental representation[20] gives exactly the energy of 't Hooft-type electic flux. Consequently, for *finite L*, the quantities K_L[7,17] and $\sigma(L)$[18] are nothing but $\Delta E(L)/L$, which only for large L is expected to give the true string tension. The 0^{++} glueball mass was determined either by the time-time correlation function for the spatial Polyakov loop in the *adjoint* representation, giving a good signal for $1 \lesssim z \lesssim 5$,[17] or for the 2×2 Wilson loops, with a good signal for larger z.[18] In Fig. 1 we represent these Monte Carlo data by plotting $\mathcal{E} = \Delta E(L)/M_L(0^{++})$ as a function of $z = M_L(0^{++})L$. Although in this case not enough data are available to verify universality, experience with the SU(2) results[6] tells us that at least for the smaller z values lattice artifacts should be small. Note also that the two data

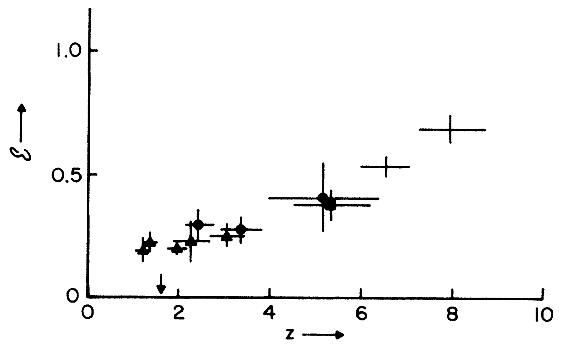

FIG. 1. Results for the energy of electric flux as a function of L, converted to dimensionless units with the 0^{++} glueball mass [$\mathcal{E} = \Delta E/M_L(0^{++})$, $z = M_L(0^{++})L$], for SU(3). Indicated is the analytic prediction for the crossover value of z (for smaller z, $\mathcal{E}$ is practically zero) and Monte Carlo data for a $N_s^3 \times N_t$ lattice. Triangles: $N_s = 4$, $N_t = 32$. Circles: $N_s = 6$, $N_t = 32$ (Ref. 17). Squares: $N_s = 6$, $N_t = 21$. Crosses: $N_s = 9$, $N_t = 21$ (Ref. 18).

points around $z = 5$, arising from different ways to calculate the glueball mass, are consistent with each other. [Let us point out that Ref. 17 uses the definition $z = M_L(2^{++})L$, at odds with previous works.[6-8,12]] A more accurate way to determine the point of crossover is to use the spatial Polyakov loop in the fundamental representation as a signal, in a fashion similar to extracting the deconfining temperature.[21] To be more precise, one tests with a given lattice for the value of g_0 where metastablity sets in and calculates $M_L(0^{++})$ for that value of g_0. This gives the crossover value of z, which should not depend on the lattice size N_s (provided $N_t \gg N_s$). It is in the domain close to metastability that evaluation of the energy of electric flux becomes numerically impractical. It is important to observe, however, that there is a marked difference from the deconfining transition.[21] In our case the crossover, although rapid, is smooth and the expectation value of the absolute value of the spatial Polyakov loop remains reasonably close to one over the crossover; the onset of tunneling causes the phase of the spatial Polyakov loop to fluctuate rapidly. The effective potential which describes the tunneling[6] changes very little though, because of the smallness of the coupling constant. Consequently, one should classify the domain for z a little larger than 1.6 as a perturbative domain. Figure 1 shows that there is a discrepancy between the Monte Carlo data, which has a data point for z as small as 1.2 ± 0.2, and our analytic prediction which implies that $\mathcal{E}$ is essentially zero for $z \lesssim 1.6$. This might be caused by the inaccuracy of the semiclassical approximation for $g^2 \sim 0.4$, and calculations going beyond the semiclassical approximation, but with use of the one-loop effective Hamiltonian,[15] are in progress. However, as pointed out earlier, for g even a little smaller than g_c, the Monte Carlo calculations in Ref. 17 are not at zero temperature compared to the energy of electric flux. If indeed tunneling sets in for $z \sim 1.6$, it could be that the two data points (see Fig. 1) for $z \lesssim 1.6$ measure finite-temperature artifacts instead of the energy of electric flux. Since details on the Monte Carlo analysis of Ref. 17 are not yet available, we cannot assess these problems in the numerical part of the analysis.

Only for substantially larger values of z will one approach the truly nonperturbative domain and for four reasons it is expected to be associated with a crossover around $z \sim 5$[6,22]: (i) This is the value one would obtain from the finite temperature intuition by putting $L = 1/T_c$; (ii) it is the value where the square-root singularity in the Nambu-Goto expression for $\mathcal{E}$ occurs; (iii) a second crossover is expected because of the tunneling by "normal" instantons; and (iv) the separation of zero and nonzero momentum modes is likely to fail when the glueball mass is of the order of energy of a typical nonzero momentum state ($E = 2\pi/L$), which hence, is expected to occur around $z = 2\pi$. Let us stress that this second crossover is also *not* a phase transition, because of the finite volume. The existence of a second crossover would also mean that a naive extrapolation beyond $z = 5$, especially for the ratio $M(0^{++})/M(2^{++})$, which was found to be larger than 1 for small z,[17] is very likely not possible. Surely it is no surprise that $M(0^{++}) > M(2^{++})$ for a small volume, first of all for $z < 1$ it has been rigorously established[12,13] that $M(0^{++})/M(2^{++}) \simeq 1.2$, second our intuition that higher-angular-momentum states should be heavier is not valid when rotational invariance is badly broken, as is the case in a small cubic volume. The tunneling for z around 1 is not expected to alter the ratio dramatically,[6] and indeed for SU(2) the Monte Carlo data[17] connect very well with the analytic result.[12] But it is not at all unlikely that around $z = 5$ the ratio $M(0^{++})/M(2^{++})$ (already close to 1 at $z \simeq 4.5$) has a crossover, or continues more slowly, towards smaller values, making the 0^{++} glueball the lightest as expected on the basis of angular momentum[23] and consistent with strong-coupling expansion.[24] Arguments for a crossover of the $M(0^{++})/M(2^{++})$ ratio around $z = 5$ were also given by Michael.[25]

For the SU(3) data,[17,18] the same comments hold. However, it is worth pointing out that the recent perturbative computation for $M(0^{++})/M(2^{++})$ by Weisz and Ziemann[13] yields a value of approximately 1.2, which deviates considerably from the value of 1.6 ± 0.2 at $z = 1.2 \pm 0.2$.[17] Of course, it is not ruled out that this discrepancy is due to the tunneling phenomena.

In conclusion, recent Monte Carlo calculations in a finite volume start to make contact with analytic calculations. Work is in progress to establish analytic results in a domain up to, say $z = 2$, with methods essentially applicable to both SU(2) and SU(3).[15] This should allow us to determine the crossover value of z more accurately. Although this exciting recent progress is still far removed from complete control in the infinite volume, it is likely to give this desired control in a small volume, with interesting information on the approach towards the infinite volume.

We are very grateful to Peter Weisz for sharing with us the SU(3) perturbative results at a very early stage and allowing us to use those results prior to publication. P.v.B. would also like to thank Martin Lüscher, Poul Olesen, Jan Ambjøn, Gerard 't Hooft, and the theory group of CERN for warm hospitality at DESY, the Niels Bohr Institute, the Institut voor Theoretische Fysica at Utrecht, and CERN, and thanks them and F. Gutbrod, Morton Laursen, Chris Michael, and I. Muzinich for discussions. Finally, we thank Jan Smit for useful comments on the manuscript. This work was supported in part by National Science Foundation Grants No. DMS-84-05661 and No. PHY-85-07627.

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