The group theory of twist eating solutions

by Bert van Geemen\textsuperscript{1} and Pierre van Baal\textsuperscript{2}

\textsuperscript{1} Department of Mathematics, Budapestlaan 6, P.O. Box 80.010, 3508 TA Utrecht, the Netherlands
\textsuperscript{2} Institute for Theoretical Physics, State University of New York at Stony Brook, Long Island, New York 11794, USA

Communicated by Prof. G. 't Hooft at the meeting of September 30, 1985

ABSTRACT

In this note we show the relation between solutions to the equation $[\Omega_\mu, \Omega_\nu] = \exp \left( \frac{2\pi in_{\mu\nu}}{N} \right) I$ and the representations of the Heisenberg group, where $\mu = 1, 2, \ldots, 2g$ and $\Omega_\nu \in SU(N)$. We construct all its irreducible representations and whence all solutions to the above equation for arbitrary $g$. We give a criterion for existence and uniqueness of solutions.

1. THE SETTING OF TWIST EATING SOLUTIONS

Twisted gauge fields on the hypertorus, both in the continuum and on the lattice posed an interesting mathematical problem, namely finding $SU(N)$ matrices $\Omega_\mu$ (called twist eating solutions), such that:

$$
\begin{align*}
[\Omega_\mu, \Omega_\nu] &= \Omega_\mu \Omega_\nu \Omega_\mu^{-1} \Omega_\nu^{-1} \\
&= \exp \left( \frac{2\pi in_{\mu\nu}}{N} \right) I
\end{align*}
$$

$n$ is called the twist tensor, it is skew symmetric with integer entries mod $N$. The index $\mu$ runs from 1 to $2g$ (the dimension of space-time; odd dimensions need not be considered separately). For details and further references see [1, 2] where the full solution of this problem for $g \leq 2$ was found.

By means of a $SL(2g, \mathbb{Z})$ transformation $X$, we can always transform $n$ to its standard form:
\[ (2) \quad n^{(0)} = \begin{pmatrix} \oplus & e_1 \\ -e_1 & \cdot \cdot \cdot \\ \cdot \\ \cdot \\ -e_8 & \oplus \end{pmatrix} \]

where \( e_1, e_2, \ldots, e_8 \) and \( n = X^T n^{(0)} X \). If \([U_{\mu}, U_{\nu}] = \exp (2\pi i n^{(0)} / N) f\), then equation (1) is solved by

\[ (3) \quad \Omega_{\mu} = \prod_{\nu} U_{\nu}^{N_{\nu}}. \]

From now on we assume \( n \) to be in the form (2).

This form is not unique since we can add a multiple of \( N \) to \( n_{\mu\nu} \). However, since eq. (3) can be inverted [2], the specific choice of \( n^{(0)} \) is irrelevant.

Previously we established that solutions exist if \( n_{\mu\nu} \) is an integer, for \( g = 1, 2 \) (where \( Pf \) is the Pfaffian which is a square root of the determinant of an even dimensional skew symmetric matrix; \( Pf(n^{(0)}) = -\prod_{i=1}^{g} (-e_i) \)). In this note we will show that for arbitrary \( g \), \( Pf(n / N) N \in Z \) is a necessary but not a sufficient condition for existence of solutions to eq. (1), consistent with the remark in [1] that \( Pf(n) = 0 \mod N \) is not a sufficient condition.

It was also shown for \( g \leq 2 \) that the solution to equation (1) is unique up to a similarity transformation and multiplication with an element of the centre of \( SU(N) \) if \( g \geq 2 \):

\[ (4) \quad \gcd \left( n_{\mu\nu}, Pf \left( \frac{n}{N} \right) N, N \right) = 1. \]

For \( g > 2 \) this is a sufficient, but not a necessary condition. We will show that uniqueness up to a similarity transformation means that the matrices \( U \) generate an irreducible representation of the Heisenberg group, in which case there are \( N^{2g-1} \) \( SU(N) \)-inequivalent solutions.

It is clear that the group \( G \) generated by \( U \) has the property that its commutator is in the centre of the group \( G \) (\( G \) is therefore nilpotent). This is typical of the so-called Heisenberg group. (Indeed also in physics the Heisenberg commutation relation between coordinates and their canonical momenta satisfy the same property).

In the next section we will describe the relevant Heisenberg group and its Schrödinger representation. In section 3 all representations of the Heisenberg group are classified. In section 4 these results are used to construct all solutions to (1) and give the appropriate criteria for existence and uniqueness, based on \( n \).

\footnote{For integer \( p \) and \( q \) the symbol \( p \mid q \) means that \( p \) divides \( q \).}

\footnote{iff = if and only if, \( g.c.d. = \) greatest common divisor.}
2. THE HEISENBERG GROUP AND ITS CANONICAL REPRESENTATION

In general one can describe the Heisenberg group by the following properties. Let $K$ be an (additive abelian) group, $C^* = C - \{0\}$ the multiplicative group of complex numbers and $K^*$ the dual of $K$, i.e., $K^*$ is the (additive) group of homomorphisms $f: K \to C^*$. We will denote its elements by $x^*$. The Heisenberg group is given by $H = C^* \times K \times K^*$, with a product defined by:

\[
\begin{cases}
H \times H \to H : (t, x, x^*) \cdot (s, y, y^*) \\
= (t sy^*(x), x + y, x^* + y^*).
\end{cases}
\]

For completeness let us mention the following properties

\[
\begin{align*}
\cdot x^*(a + b) &= x^*(a) \cdot x^*(b) \\
(x^* + y^*)(a) &= x^*(a) \cdot y^*(a) \\
(t, x, y^*)^{-1} &= (t^{-1} y^*(x), -x, -y^*) \\
[(t, x, y^*), (s, u, v^*)] &= (v^*(x) \cdot y^*(a)^{-1}, 0, 0).
\end{align*}
\]

In principle we can restrict ourselves to the subgroup

\[\{t \in C^* \exists x \in K, y^* \in K^*: t = y^*(x)\} \text{ of } C^*.
\]

When $K$ is finite this makes $H$ a finite group, which is convenient for finding all its representations. In fact when $K$ is finite we must have that $y^*(x)$ is a root of unity for all $x$ and $y^*$.

We will now introduce the Heisenberg group $H(\delta)$, of type $\delta$, where:

\[
\begin{cases}
\delta = (d_1, d_2, \ldots, d_k) \\
d_i \in \mathbb{N}, d_i |d_2| \ldots |d_k|.
\end{cases}
\]

The length or norm of $\delta$ is given by:

\[|\delta| = d_k.
\]

To $\delta$ we associate the group $K(\delta)$:

\[
K(\delta) = Z_{d_1} \times Z_{d_2} \times \ldots Z_{d_k}
\]

where $Z_n$ stands for $\mathbb{Z}/n\mathbb{Z}$, being the additive group of integers modulo $n$. To distinguish this from the unimodular group of the $n$-th roots of unity the latter will be denoted by $\mu_n$, which is a multiplicative group:

\[
\begin{cases}
\mu_n = \{1, e^{2\pi i/n}, e^{4\pi i/n}, \ldots, e^{2\pi i(n-1)/n}\} \\
= \{t \in C^* | t^n = 1\}.
\end{cases}
\]
The centre of $SU(N)$ is also denoted by $\mu_N$. We define $H(\delta)$ by:

$$H(\delta) = \mu_{g_1} \times K(\delta) \times K(\delta)^*.$$  

Let $q : \mathbb{Z}^N \to K(\delta)$ be the canonical projection.

Define an isomorphism $K(\delta) \to K(\delta)^*, x \mapsto * (x)$ by:

$$\begin{align*}
    \ast(q(x))q(y) = \prod_{j=1}^{N} \exp \left( \frac{2\pi i x_j y_j}{d_j} \right) \\
    \text{(so that on the group $K(\delta)$ it becomes)} \\
    \left( \exp \left( \frac{2\pi i \sum_k v_k x_k - y_k H_k}{d_k} \right), 0, 0 \right).
\end{align*}$$

We will next construct the canonical representation of $H(\delta)$, called the Schrödinger-representation, which is denoted by $\sigma_\delta$. The representation space will be the $C$-vector space of functions $f : K(\delta) \to C$. Define an action of $H(\delta)$ on this vector space of dimension $N = \prod_{j=1}^{N} d_j$ by:

$$\left( \sigma_\delta(t, x, y^*) (f) \right)(z) = \exp (z f(x + z)).$$

Let us construct explicitly the $N$-dimensional unitary matrix representation of $\sigma_\delta$. A $C$-basis of $\text{Func} (K(\delta) \to C)$ is given by

$$f_{\delta}(x) = \begin{cases} 1 & x = a, \quad x, a \in K(\delta) \\ 0 & x \neq a \end{cases}$$

or equivalently

$$f_{\delta}(x) = \delta_{x, y^*}.$$  

In an obvious notation: $f_{\delta}(x) = \delta_{x, y^*}$.

It is easy to check that the matrix $\sigma_\delta(t, x, y^*)_{ab}$ is defined by:

$$[\sigma_\delta(t, x, y^*)]_{ab} = \sum_{j \in K(\delta)} \sigma_\delta(t, x, y^*)_{ab} f_{\delta},$$

i.e.

$$\begin{align*}
    \left( \sigma_\delta((t, x, y^*) \cdot (s, u, v^*)) (f_{\delta}) \right) = \\
    \sum_{a \in K(\delta)} \left[ \sigma_\delta(t, x, y^*) \right]_{a, a} \sigma_\delta(s, u, v^*) \left[ \sigma_\delta(t, x, y^*) \right]_{a, b} f_{\delta}.
\end{align*}$$

To write down the matrices we use:

$$\left( (t, q(x), \ast(q(y))) = \prod_{j=1}^{N} (1, 0, 0)^{y_j} \cdot \prod_{j=1}^{N} (1, 0, 0)^{y_j} \right),$$

with

$$(u_j)_k = \delta_{j, k}.$$
If we introduce:

\[ U_{z-j+1} = \hat{U}_j = \sigma_\delta(1, 0, u^*_j), \quad U_{2k-j+1} = \hat{U}_{k+j} = \sigma_\delta(1, u_j, 0) \]

we have (from now on we ignore the difference between \( x \) and \( q(x) \)):

\[ \sigma_\delta(t, x, y) = t \prod_{j=1}^{\frac{N}{d_k}} \hat{U}_{j}^{y_j} \prod_{j=1}^{\frac{N}{d_k}} \hat{U}_{k+j}^{y_j}. \]

Explicitly we have:

\[
\begin{align*}
(U_{z-k+1})_{a,b} &= \exp\left(\frac{2\pi i a_k}{d_k}\right) \delta_{a,b} \\
(U_{2k-z+1})_{a,b} &= \delta_{a+a_k, b}.
\end{align*}
\]

Or as tensor products we can write:

\[
\begin{align*}
U_{z-k+1} &= 1_d \otimes \cdots \otimes Q_k \otimes \cdots \otimes 1_d, \\
U_{2k-z+1} &= 1_d \otimes \cdots \otimes P_k \otimes \cdots \otimes 1_d,
\end{align*}
\]

where \( I_n \) is the \( n \)-dimensional identity and \( Q_n, P_n \) are the twist matrices satisfying

\[ [P_n, Q_n] = e^{2\pi i n/\ell, 1_n} \]

with

\[ Q_n = \text{diag}(1, e^{2\pi i /n}, \ldots, e^{2\pi i (n-1)/n}) \]

\[ P_n = \begin{pmatrix}
0 & 1 & 0 & \cdots \\
0 & 0 & \ddots & \cdots \\
\vdots & \ddots & \ddots & 1 \\
1 & \cdots & \cdots & 0
\end{pmatrix}.
\]

It is an easy exercise to show that this representation is irreducible. Furthermore we have that \( U_\mu \) satisfy eq. (2) with:

\[ e_j = -\frac{N}{d_{z-j+1}}, \quad N = \prod_{k=1}^{s} d_k. \]

Note that \( Pf(n/N) = -N^{-1} \); therewith we verified for this particular representation: \( Pf(n/N)N \) is integer and g.c.d. \( (n_\mu, N, Pf(n/N)N) = 1 \).

3. ALL REPRESENTATIONS OF THE HEISENBERG GROUP \( H(\delta) \)

In order to find all representations, we can invoke the well known result [3] due to Frobenius for a finite group \( H \):

\[ \sum_\delta (\dim \delta)^2 = 0(H) \]
where the summation runs over the irreducible representations \( \varrho \) and \( \mathfrak{g}(H) \) is the order (number of elements) of \( H \). We have:

\[
(26) \quad \mathfrak{g}(H(\delta)) = |\delta|(\prod_{j=1}^{r} d_j)^2.
\]

Denote by \( C(\delta) \) the centre of \( H(\delta) \)

\[
(27) \quad C(\delta) = \{ (t, 0, 0) | t \in \mu_{|\delta|} \}
\]

and let \( \varrho \) be some irreducible representation of \( H(\delta) \) on a vector space \( V \) (\( \dim \varrho = \dim V \)). Then by Schur's lemma:

\[
(28) \quad \varrho(\mathcal{I}) = \lambda_\varrho(\mathcal{I}) \mathcal{I}, \quad \forall \mathcal{I} \in C(\delta)
\]

with \( \mathcal{I} \) the identity on \( V \) and \( \lambda_\varrho(\mathcal{I}) \in \mathbb{C} \).

Obviously \( \lambda_\varrho : C(\delta) \to \mathbb{C} \) is a homomorphism which we will call the central character (of \( \varrho \)). Note that if two irreducible representations \( \varrho \) and \( \varrho' \) satisfy \( \lambda_\varrho \neq \lambda_{\varrho'} \), they cannot be equivalent. Since \( C(\delta) \) is cyclic of order \( |\delta| - 1 \) we must have:

\[
(29) \quad \lambda_\varrho((t, 0, 0)) = t^m, \quad 0 \leq m \leq |\delta| - 1.
\]

The following, so-called twisted Schrödinger representation has this central character:

\[
(30) \quad \{ \{ \sigma_\varrho(m)(t, x, y^\ast)(f) \}(z) = [t \cdot y^\ast(z)]^m f(x + z).
\]

Following the same steps as in the previous section we find the matrix representation of \( \sigma_\varrho(m) \) by:

\[
(31) \quad \begin{cases} 
U_{z-k+1} = \sigma_\varrho(m)(1, 0, u^\ast_z) = 1_{d_1} \otimes \cdots \otimes Q^m_{d_k} \otimes \cdots \otimes 1_{d_s} \\
U_{z-k+1} = \sigma_\varrho(m)(1, u_k, 0) = 1_{d_1} \otimes \cdots \otimes P_{d_k} \otimes \cdots \otimes 1_{d_s}.
\end{cases}
\]

An invariant subspace \( W \) is found by considering each tensor component separately. For the \( k \)-th component we have \( (z = (1, 1, 1 \ldots 1) \in \mathbb{C}^{d_k}) \)

\[
(32) \quad W_k = \langle z, Q^m_{d_k} z, Q^{2m}_{d_k} z, \ldots, Q^{m(d_k-1)}_{d_k} z \rangle.
\]

Using the explicit diagonal form of \( Q_{d_k} \) in eq. (23) we find 3:

\[
(33) \quad \dim W_k = d_k / \text{g.c.d.} (m, d_k).
\]

On the other hand, when \( \text{g.c.d.} (m, d_k) = 1 \) the eigenvalues of \( Q^m_{d_k} \) are a permutation of those of \( Q_{d_k} \), which implies that there is no non-trivial invariant subspace.

Let us introduce the notation:

\[
(34) \quad \begin{cases} 
\pi = (\rho_1, \rho_2, \ldots, \rho_s), \quad p_k = \text{g.c.d.} (m, d_k) \\
y = (c_1, c_2, \ldots, c_s), \quad c_k = d_k / p_k.
\end{cases}
\]

\[5 \text{ For } m = 0 \text{ we define g.c.d. } (m, d_k) = d_k.\]
From eq. (7) we have that \( p_k = 1 \) implies that all other \( p_k \) equal 1. We conclude: \( \sigma_\delta(m) \) is irreducible iff \( m \) and \( |\delta| \) are relatively prime.

Assume now that \( \varphi \) is an irreducible representation of \( H(\delta) \) with central character \( \lambda(\delta) = t^m \) and with \( \text{g.c.d.} (m, |\delta| = d_\delta) \neq 1 \). Define \( \pi, \gamma \) as above and

\[
C(\delta, m) = (\mu(\delta), \gamma K(\delta), (\gamma K(\delta))^*)
\]

where we used the shorthand notation:

\[
yK(\delta) = c_1 Z_{d_1} \times c_2 Z_{d_2} \times \ldots \times c_\delta Z_{d_\delta}.
\]

Note that elements of \( \varphi(C(\delta, m)) \) commute with \( \varphi(H(\delta)) \), hence by Schur's lemma \( \varphi(C(\delta, m)) \) consists of scalar multiples of the identity. Therefore \( \varphi \) restricted to \( C(\delta, m) \) is given by \((a, b) \in K(\pi)\) and \( \gamma \cdot x = (c_1 x_1, \ldots, c_\delta x_\delta) \in K(\delta)):

\[
\varphi(t, y \cdot x, y \cdot y^*) = t^m \chi_{a,b}(x, y)I
\]

with

\[
\chi_{a,b}(x, y) = \exp \left( 2\pi i \sum_k \frac{x_k a_k + y_k b_k}{p_k} \right).
\]

Clearly two irreducible representations with different \( m, a, b \) would be inequivalent.

Let us now consider the case where \( a = b = 0 \). Then \( \varphi \) is trivial on the subgroup \( C_0(\delta, m) \) of \( H(\delta) \):

\[
C_0(\delta, m) = ((\mu(\delta))^{p_1}, \gamma K(\delta), (\gamma K(\delta))^*)
\]

Therefore \( \varphi \) factors over \( C_0(\delta, m) \):

\[
\begin{array}{ccc}
H(\delta) & \xrightarrow{\varphi} & V \\
\downarrow & & \downarrow \varphi \\
H(\delta)/C_0(\delta, m) & & \\
\end{array}
\]

\( \varphi \) is a faithfull representation of \( H(\delta)/C_0(\delta, m) \).

On the other hand \( H(\delta)/C_0(\delta, m) \) is isomorphic to \( H(\gamma) \), which is most easily seen by defining a homomorphism \( \phi^\gamma : H(\delta) \to H(\gamma) \) whose kernel coincides with \( C_0(\delta, m) \):

\[
\phi^\gamma : (t, x, y^*) \mapsto (t, x^\gamma, \psi(y), \psi^*(y^\gamma))
\]

with

\[
\left\{
\begin{array}{l}
x_k = x_k \mod c_k \\
\psi_k(x) = (p_k^{-1} \pi) x_k \\
\psi^*(y^\gamma)(x) = \exp \left( 2\pi i \sum_k \frac{y_k x_k}{p_k d_k} \right).
\end{array}
\right.
\]

45
(Note that: i) $\psi^*(_v \psi) = _v \psi$ in an obvious way, however one should be careful with this notation when one identifies $q(x)$ with $x$, ii) $\psi : K(\delta) \to K(\gamma)$ is surjective because $\rho_{\pi}^{-1}[\pi]$ and $c_i$ are relatively prime iii) $\psi(\gamma \cdot x) = 0$ and $\psi^*(_v \psi) = 0$.)

It is now obvious that an irreducible representation with central character $i^n$ is given by: $\phi^\delta_r = \sigma_r(m/[\pi]) \circ \phi^\delta_r$

\[
\begin{array}{c}
\phi^\delta_r \\
\downarrow \\
H(\gamma)
\end{array}
\xrightarrow{\sigma_r(m/[\pi])}
\begin{array}{c}
\phi^\delta_r \\
\downarrow \\
H(\delta)
\end{array}
\xrightarrow{\phi^\delta_r}
\begin{array}{c}
V
\end{array}
\]

(43)

Explicitly (with $f \in \text{Func}(K(\gamma), \mathbb{C})$, $z \in K(\gamma)$):

\[
(\rho(t, x, y^*) f)(z) = (1 \otimes \psi^*(_v \psi)(z))^{m/[\pi]} \bar{f}(\psi(x) + z).
\]

Finally for $a, b \neq 0$ we can extend $X_{a,b}$ uniquely to a 1-dimensional representation of $H(\delta)$, with (necessarily) $m = 0$:

\[
(45) \quad \tilde{X}_{a,b}(t, x, y) = \exp \left( 2\pi i \sum \frac{x_k a_k + y_k b_k}{d_k} \right).
\]

The product representation:

\[
(46) \quad \phi = \tilde{X}_{a,b} \cdot \phi^\delta_r
\]

is therefore an irreducible representation for each $m$, $a$ and $b$.

We have:

\[
(47) \quad \begin{cases}
\sum_{m=0}^{[\pi]} \sum_{a,b \in \mathbb{R}(\pi)} \text{dim} (\tilde{X}_{a,b} \cdot \phi^\delta_r)^2 \\
= \sum_{m=0}^{[\pi]} (\prod_{k=1}^{\delta} p_k)^2 \left( \prod_{k=1}^{\delta} c_k \right)^2 = [\delta] \prod_{j=1}^{\delta} d_j^2.
\end{cases}
\]

Using (25) and (26) we see that we have found all irreducible representations. They are in essence all twisted Schrödinger representations.

4. BACK TO THE TWIST

Let us first write down the matrices $U_{\rho}$ for $\phi^\delta_r$:

\[
(48) \quad \begin{cases}
U_{x-k+1} = \phi^\delta_r(1, 0, u^*_{k}) = 1 \otimes \ldots \otimes \rho_{\pi}^{i(p_0/\pi)} \otimes \ldots \otimes 1
\\
U_{x-k+1} = \phi^\delta_r(1, u_k, 0) = 1 \otimes \ldots \otimes \rho_{\pi}^{i(p_0/\pi)} \otimes \ldots \otimes 1
\end{cases}
\]

which satisfy eq. (2) with:

\[
(49) \quad e_{x-k+1} = -N \frac{m/p_k}{d_k} = -\frac{Nm}{d_k}, \quad N = \prod_{k=1}^{\delta} c_k^2 = N_{\text{tr}}.
\]
Furthermore we have:

\[ Pf\left(\frac{n}{N}\right)N = -\prod_{i=1}^{r} \left(\frac{m}{p_k}\right) \in \mathbb{Z}. \]

Clearly \(m_k = m/p_k\) is relatively prime to \(c_k\). Since \(c_i|c_k\) for \(i < k, m_k\) is also relative prime to \(c_i\).

For \(g \leq 2\) this is easily seen to imply g.c.d. \((n_{irr}, NPf(n/N), N) = 1\). However, the following example, due to Coste, shows that it is not a necessary condition for irreducibility. Take:

\[ \begin{cases} g = 3, N = 2^2 \cdot 7^3 \\ e_3 = 4e_2 = 4e_1 = 2^3 \cdot 3 \cdot 7^2. \end{cases} \]

For this, one explicitly verifies that:

\[ \text{g.c.d. } \left( n_{irr}N, NPf\left(\frac{n}{N}\right), N \right) = 2 \neq 1, \]

but that it nevertheless admits a solution which is based on the irreducible representation \(\sigma_6(m)\) with:

\[ \delta = (7, 28, 28), \quad m = 6. \]

To show that \(N \cdot Pf(n/N) \in \mathbb{Z}\) is not a sufficient condition\(^4\) we give the next example, also provided by Coste:

\[ \begin{cases} g = 3, N = 2^2 \cdot 3^6 \\ e_3 = 2^4e_2 = 2^4e_1 = 2^43^4. \end{cases} \]

This yields \(Pf(n/N)N = 1\), but it cannot allow for a twist eating solution. One way to see this is to use the well known result, that

\[ U(k) = \prod_{\mu=1}^{n} U_\mu^{k_\mu} \]

are independent \(N \times N\) matrices for \(0 \leq k_\mu < N_\mu\) if a solution does exist. Where:

\[ N_i = N_{i+1} = e_2 \times N_{i+1}. \]

This is an easy generalization of the well known result for \(g = 1\) and 2, see e.g.\([2]\).

Therefore we have at least \(\prod_{\mu} N_\mu = N_{irr}^2\) independent \(N \times N\) matrices. There can be no more than \(N^2\), so that necessarily

\[ N_{irr} \leq N. \]

\(^4\) Integrity of \(Pf(n/N)N\) in general depends on the choice of \(n\); i.e. it depends on its mod \(N\) freedom.
For the case of eq. (54) this bound is easily seen to be violated, hence $Pr(n/N)N \in \mathbb{Z}$ is not sufficient for the existence of solutions to eq. (1).

Since for given $e_j$ and $N$, $c_j$ and $m_j$ are fixed, one can always find $d_j$ and $m$ such that eq. (7) is satisfied. The simplest choice for $m$ is the smallest common multiple of all $m_j$, which equals $m_1$ (see below eq. (50)).

Therefore if $N$ is a multiple of $N_{rr}$ we can write down the following solution:

(58) \[ U_\mu = A_\mu \times U_\mu^{irr}, \]

where: $A_\mu = \text{diag}(a_1^{(1)}(\mu), \ldots, a_\mu^{(N/N_{rr})}) \in U(1)^{N/N_{rr}}$ and $U_\mu^{irr} \in U(N_{rr})$ coming from $\sigma_{\mu}(m)$ (see eq. (48)).

We claim that up to a similarity transformation this is the only possible type of solution. Suppose that $U_\mu$ is a solution and define:

(59) \[ \omega_\mu = U_\mu^{N_{rr}}, \]

with $N_\mu$ defined in (56) satisfying $Z_{irr}^{N_\mu} = 1$. This implies that the $\omega_\mu$ can be simultaneously diagonalized, with $U_\mu$ block-diagonal:

(60) \[ [\omega_\mu, \omega_\nu] = [\omega_\mu, U_\nu] = 1. \]

Working in this diagonal gauge one easily verifies that:

(61) \[ \tilde{U}_\mu = A_\mu^{-1}U_\mu, \quad A_\mu^z = \omega_\mu \]

satisfy the same equation as $U_\mu$, but such that $\tilde{U}_\mu = U_\mu^{N_{rr}} = 1$. Consequently, the $\tilde{U}_\mu$ give a representation of the Heisenberg group. With the help of the previous two sections there is no other possibility for $\tilde{U}_\mu$ then to be the direct sum of irreducible representation of dimension $N_{rr}$. And hence $N$ has to be a multiple of $N_{rr}$ and $U_\mu$ is of the form of eq. (58). The dimension of the solution manifold to equation (1) up to a similarity transformation is $(N/N_{rr})-1$.

Given a solution we therefore find:

(62) \[ Pr\left(\frac{n}{N}\right)N = -\left(\prod_{k=1}^{s} m_k\right) \cdot \frac{N}{N_{irr}} \in \mathbb{Z} \]

and

(63) \[ e_{x-i+1} = m_N^{d_j} \cdot \frac{N_{rr}}{m_j} \prod_{k \neq i} c_k, \]

whence:

(64) \[ (N/N_{irr}) \quad \text{g.c.d.} \quad \left( n_{irr}, Pr\left(\frac{n}{N}\right)N, N \right). \]

One easily verifies that for $g = 1$ and 2, g.c.d. $(n_{irr}, Pr(n/N)N, N) = N/N_{irr}$ and the dimension of the solution manifold confirms with previous results [1].

\( \text{For } g = 1 \text{ and } 2, \text{ g.c.d. } (N_{irr}, \prod_{k=1}^{g} m_k) = 1, \text{ so that } Pr(n/N)N \in \mathbb{Z} \text{ implies that } N \text{ is a multiple of } N_{irr}. \)
We also see that g.c.d. \((n_{\text{irr}}, P(n/N)) = 1\) implies \(N = n_{\text{irr}}\), which implies uniqueness of the solution. Since \(n_{\text{irr}}\) fixes the dimensionality of the solution manifold, we indirectly see that in the case that \(N\) is a multiple of \(N_{\text{irr}}\) for a specific choice of \(n\), \(N_{\text{irr}}\) does not depend on this choice (because eq. (3) is invertible [2].)

In conclusion, twist eating solutions exist iff \(N\) is a multiple of \(N_{\text{irr}}\) (for which \(P(n/N)N = 0\) is only a necessary condition). A solution is unique up to a gauge and \(Z_N\)-factors iff \(N = N_{\text{irr}}\) iff it corresponds to an irreducible representation (for which g.c.d. \((n_{\text{irr}}, P(n/N))N = 1\) is only a sufficient condition), in that case (48) gives an explicit solution for eq. (2), which through a simple rescaling by a phase factor can be chosen in \(SU(N)\)

\[
\left(\text{replace } Q_n \text{ by } Q_n = \exp \left( -\frac{\pi i (n-1)}{n} \right) \cdot Q_n \text{ and similar for } P_n \right)
\]

Multiplication with a phase factor is indeed the only freedom we have, which for \(SU(N)\) reduces to multiplication with elements of the centre of \(SU(N)\) (isomorphic to \(\mu_N\)). Using the fact that

\[
\text{g.c.d. } (m|\pi^{-1}, c_k) = \text{g.c.d. } (|\pi^{-1}, c_k) = 1
\]

we have that \(\lambda \cdot Q_k^{m/\pi} \) and \(\lambda \cdot P_k^{\pi/\pi} \) are equivalent to \(Q_k^{m/\pi} \) and \(P_k^{\pi/\pi} \) if \(\lambda \in \mu_N\). Therefore all inequivalent solutions to equation (2) are given by (see (48)) \(\lambda_k U_{d-k} \) and \(v_k U_{2d-k} \) with \(\lambda_k, v_k \in \mu_N/\mu_d\). Hence there are

\[
\prod_{k=1}^{d} (\mu_N/\mu_d)^2 = N^{2g-1}
\]

inequivalent solutions.

5. CONCLUSIONS

In this note new results concern twist eating solutions for more than four dimensions; one might think of applications for TEK-models in the \(d \to \infty\) limit, where \(d\) is the dimension of space-time. However our main motivation was to show the underlying structure of the Heisenberg group.

ACKNOWLEDGEMENTS

We would like to thank prof. T.A. Springer for a remark which made us see the light. B.v.G. is financially supported by the ZWO project on moduli and P.v.B. by the Dutch National Science Foundation FOM (ZWO). This work was partially supported by NSF grant PHY 81-09110 A-03. One of us (P.v.B.) is also very grateful to Antoine Coste for the counter examples to the \(g = 2\) criteria when extended to \(g \geq 2\).

REFERENCES

1. Baal, P. van - Communications in Mathematical Physics, 92, 1 (1983).

49