Symmetries of the 4×4 periodic Lattice

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Abstract

The symmetries of a 4×4 periodic square lattice are related to those of a 4–dimensional hypercube. The same holds for d–dimensional (hyper)cubical lattices with periodicity 4 and the $2d$-dimensional hypercube.

Consider the following arrangment of the integers 0 – 15.

15	14	12	13	15
11	10	8	9	11
3	2	0	1	3
7	6	4	5	7
15	14	12	13	15

It seems a fairly arbitrary distribution of the 16 numbers which is made periodic by adding a row and a column. By closer inspection one can discover the rules:

- ⋆ the second row follows from the top row by subtracting 4
- ⋆ the next row follows by subsequently subtracting 8
- ⋆ the next by adding 4
- ⋆ the last one by adding again 8
- ⋆ the second column follows from the first by subtracting 1
- ⋆ the next column by substraction of 2
- ⋆ the next by adding 1
- ⋆ the last one by adding 2

The differences in powers of 2 suggest to use the binary notation for the numbers

1111	1110	1100	1101	1111
1011	1010	1000	1001	1011
0011	0010	0000	0001	0011
0111	0110	0100	0101	0111
1111	1110	1100	1101	1111

Now the amazing thing is that every site of this lattice is surrounded by 4 points of which the binary number differs in exactly one position. This is of course due to the fact that differences coordinates of a 4–dimensional hypercube. Indeed the topology of the arrangement of the second table is exactly that of the hypercube as every point is neighbored by the points on the hypercube next to it. Consequently the topology of the periodice 4×4 lattice is that of the 4–dimensional hypercube.

This has consequencies for the symmetries of the periodic lattice. On face value the periodic lattice has 16 translational invariances, 2 reflections and 1 interchange of the axis, which make in total 128 symmetry operations. On the other hand the hypercube has 24 permutations of the dimensional directions and 4 reflections, which make a total of 384 symmetry operations. Therefore there must be a hidden symmetry of 3 operations in the periodic lattice.

It is fun to make these hidden symmetries visible. The cyclic permutation of 3 axis of the hypercube form a subgroup of order 3. Let us therefore consider a cyclic permutation of the first 3 components $x, y, z \rightarrow z, x, y$. So take e.g. point #10 or its binary 1010 and carry out the motion such that it becomes 1100 or point #12. Thus the transformation puts point #10 on the position of the present point #12 and so on for all the points. Clearly the points #0, #1, #14 and #15 are invariant since their first 3 binary digits are equal. All the others change position and the following tableau emerges.

15	14	6	7	15
13	12	4	5	13
9	8	0	1	9
11	10	2	3	11
15	14	6	7	15

This configuration cannot be reached by a combination of the standard lattice symmetry operations. One can convince oneself of this fact by for instance looking to the surrounding of the point 0. It has again as neighbors 1, 2, 4 and 8 but in different order which cannot be obtained by a local reflection, inversion or rotation. Note that all the points have the same set of neighbors as before (as it should otherwise the topology was changed) but they appear in a twisted order.

One wonders whether this is unique for the 4×4 periodic lattice. It is not. In order to find other cases where periodic lattices have the same topology of an hypercube we observe that the hypercube has as elementary loops the square and the whole structure is a knitting together of these squares. So we must look into the lattices with this structure and we come to the (hyper)cubical lattices. Next in a $2d$–dimensional hypercube each point has $2d$ neighbors; so the connection can only be between a $2d$–dimensional hypercube and a d–dimensional periodic lattice. The periodicity must be 4 lattice sites in order to match the total 2^{2d} points of the hypercube. But then the map can indeed be made. One needs for the binary representation of the 2^{2d} points in total $2d$ digits. Then arrange the lattice points in the binary representation so that in each of the d directions in the lattice 2 arbitray chosen bits change in the order: flip the first, then the second, then the first again and finally again the second. So in 4 steps one is back to original point in accordance with the periodicity 4 of the lattice.

It is an interesting question whether there are other periodical lattices which correspond to higher dimensional polygons, but the answer to this is beyond my mathematical capacity.