The Radiation Balance for a semi-gray Atmosphere

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Abstract

The equations governing the radiation balance are discussed and partially implemented.

1 Introduction

The most pressing question in the climate debate is the effect of increasing CO_2 in the atmosphere. CO_2 owes its prominence as greenhouse gas to its strong absorption in the thermal radiation range. So CO_2 influences the radiation balance and the climate reacts to these changes. This complicated problem is extensively studied but is not fully understood as the increasingly alarming predictions show. The physics of the radiation balance is the easiest part of the problem. Unfortunately very few physicists are familiar with the equations governing the balance. Therefore a simple model showing the mechanism is welcome. A gray atmosphere, which has a uniform absorption spectrum, is such a model. However, a gray atmosphere misses the important mechanism of converting the radiation from absorbing frequencies to transparant frequencies. Here we develop a model for a semi-gray atmosphere, i.e. an atmosphere that has a gray absorbing part and a transparant part.

By confining to the radiation part of the problem no conclusion about the size of the greenhouse effect can be drawn. The temperature distribution in the atmosphere is driven, but not determined, by the distribution of radiation. Secundary processes such as convection, evaporation and cloud formation react on what the radiation balance provides. Before one can study the effect of these secondary processes one has to know the stationary temperature distribution due to the influx of solar radiation.

The earth receives energy from the sun and radiates energy back into the universe. The total amounts have to be the same, otherwise the earth would cool down or warm up. The incoming radiation is mostly in the range of frequencies of visible light and the outgoing radiation is in the thermal range with much longer wavelengths. One may think of the two streams as independent. We take the incoming stream as given and concentrate on the outgoing flow. The interaction between the radiation and the atmosphere consists of two processes (elastic) scattering and (non-elastic) absorption. Scattering shifts energy from the incoming beam to diffuse radiation, but it is no redistribution of energy and has therefore no direct influence on the temperature distribution. It is only a redistribution of the angular dependence. On the other hand absorption leads to thermal re-emission which is particularly important for the outgoing radiation.

The flux of the incoming solar radiation is

$$S_\oplus = 1.369 * 10^3\,\mathrm{W/m}^2. \tag{1}$$

Part of this radiation is reflected. The albedo of the earth fluctuates around 0.3 and is, inspite of climate changes, remarkable steady [1]. So only 70% of $S_\oplus$ reaches the surface of the earth. The ratio of the cross-section of the earth and its surface is 1/4. So the surface of the earth receives

$$S_{\mathrm{in}} = 0.70 S_\oplus/4 = 239.5\,\mathrm{W/m}^2. \tag{2}$$

This is an average number. The influx at the equator is higher and at the poles lower. In Fig. 1 we plot the incomming radiation as function of the lattitude. The differences are

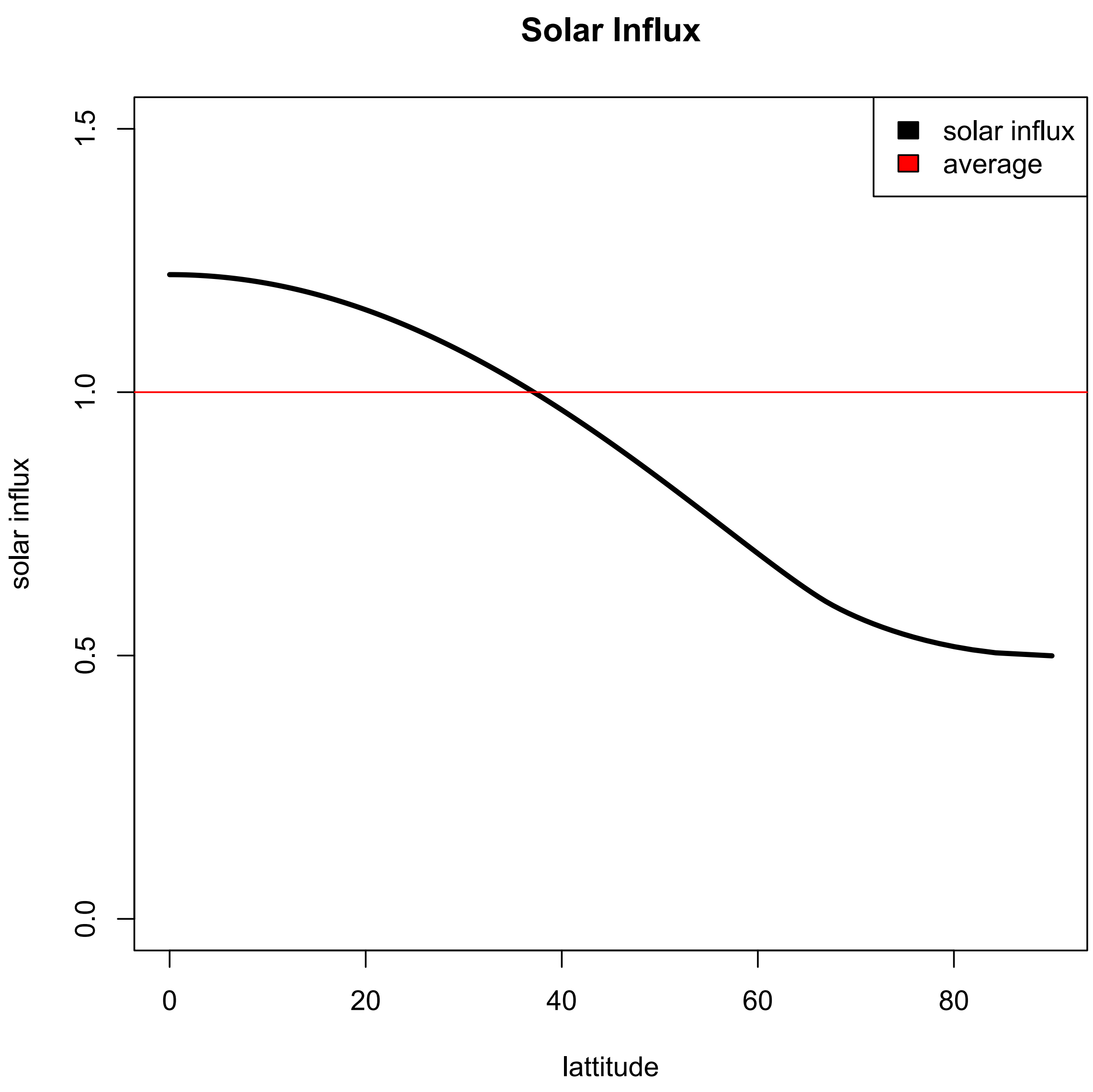

Figure 1: Yearly average of the radiation as function of the lattitude θ

large. At the equator the influx is 25% larger than average and at the poles 50% smaller than average.

The only way to get rid of the energy of the incoming solar radiation is to radiate the same amount back into space. There is no need to balance the in- and outgoing radiation at every moment and everywhere. As the radiation equation are linear it suffices to balance the radiation on the average in space and time. One may consider the two extremes, one where the earth is perfectly conducting such that the radiation temperatures are the same for all positions and one where the earth is a perfect insulator. Then the balance between and incoming and outgoing radiation has to be satisfied for every lattitude. The reality is in between. The average temperatures on the various lattitudes are quite different but not as large as the influx would give.

In order to discuss the problem of the radiative transfer we make some simplifications.

- We consider only the radiation and temperature in the *atmosphere*. Although the CO_2 density has influence on the stratosphere (stratospheric cooling), the chemical processes induced by incoming sunlight influence the stratospheric temperature, but are less important for the greenhouse mechanism in the atmosphere.

- We restrict ourselves to the radiation only. The radiative transport is fast with respect to other influences like turbulence and convective exchange in the atmosphere. Clouds are a big spoiler of the radiation balance. On the incoming side one can account for their influence by adjusting the albedo. On the outgoing side this is more difficult. We restrict ourselves to a clear sky scenario. Fortunately one can select the spectra of the outgoing radiation also from the clear sky case.

- We take the radiation as only depending on the vertical direction z in the atmosphere. The justification is that variations in the vertical vertical are much larger than in the horizontal direction.

There are three streams of relevant thermal radiation. The primary outgoing stream is the radiation from the surface of the earth. While passing through the atmosphere it is partially absorbed. The absorbed energy is released again and generates secondary streams in upward and downward direction. The basic equation for the strength of these streams is the condition that at every layer in the atmosphere the net (upward) current must be constant and equal to the incoming solar radiation. From this equation for the radiation flow we derive the temperature distribution of the atmosphere and in particular the surface temperature of the earth, which gives the magnitude of the greenhouse effect.

Before we discuss the model we first collect some data on the input of the solar radiation. Then we construct solution of the radiation balance in first approximation where $r = 1$, i.e. where the absorption is constant over the whole thermal regime. This is the so-called gray atmosphere. The physics and formulae of this special case are simple and serve as a check on the solution of general case $r < 1$, which is mathematically much more involved, but still exactly soluble. Finally we analyze how the density of CO_2 influences the parameters q and r and draw the conclusion.

2 The radiation input

The basic relation between temperature and black body radiation is the law of Stefan-Boltzmann

$$F = \sigma_{\rm SB} T^4 \tag{3}$$

Here T is the absolute temperature of the radiating substance, F the outgoing radiation flux and $\sigma_{\rm SB}$ is a universal constant with the value

$$\sigma_{\rm SB} = 5.670374419 * 10^{-8} {\rm W/m^2K^{-4}}. \tag{4}$$

$\sigma_{\rm SB}$ involves only fundamental constants like the speed of light, the Planck constant and the Boltzmann constant.

The input radiation is determined by the surface temperature of the sun $T_\odot = 5778$ K and a number of geometrical factors. The flux of solar radiation $S_\odot$ is associated with the temperature of the sun

$$S_\odot = \sigma_{\rm SB} T_\odot^4 = 5.6704 * 10^{-8} (5778)^4 = 6.319 * 10^7 \, {\rm W/m^2}. \tag{5}$$

The radius of the sun $R_\odot = 6.957 * 10^8$ m and the distance of the earth to the sun is $R_\oplus = 1.496 * 10^{11}$ m. Thus the strength of the flux at the earth is diminished to

$$S_\oplus = S_\odot (R_\odot / R_\oplus)^2 = 1.369 * 10^3 \, {\rm W/m^2}. \tag{6}$$

The earth has a cross-section for the radiation of πR_O^2 and a surface $4\pi R_O^2$. So on the average the earth receives the flux $F_\oplus/4$. Moreover the earth has an average albedo of 0.29, which means the 29% of the incoming flux is reflected and 71% reaches the surface of the earth. So the effective in-flux is So the effective in-flux is

$$S_{\rm in} = 0.71 S_\oplus / 4 = 243 \, {\rm W/m^2}, \tag{7}$$

which heats the earth. The problem is how it is transmitted through the atmosphere back into space.

A simplistic view, due to Arrhenius, is to see the earth as a black body radiator with a uniform temperature $T_{\rm out}$. According to the Stefan-Boltzmann law $T_{\rm out}$ should be

$$T_{\rm out} = (S_{\rm in}/\sigma_{\rm SB})^{1/4} = 256 \, {\rm K}. \tag{8}$$

Compared to the average real temperature of 288 K this differs 32 K, which is attributed to the greenhouse effect.

The greenhouse effect is the temperature difference between the surface of the earth and the top of the atmosphere where the air is so dilute that thermal radiation passes without noticable absorption. There the outgoing flux must equal the flux given by Eq. (7) in order that the earth keeps the same temperature. An alternative and more useful measure for the greenhouse effect is the ratio R of the temperature at the surface of the earth and the temperature at the top. This is a property of the composition of the atmosphere. The estimate of Arrhenius gives for the present value R=288/256=1.125. The fourth power R^4 gives the ratio of the intensity of the radiation at the surface and the top. Since the equation for the radiation is linear we may use R to estimate the ground temperature for different values of the top radiation.

An example of such different values of the outgoing radiation follows from the opposite of the Arrhenius assumption. Arrhenius assumes that the temperature on the earth is everywhere the same as it would be if the earth would be a perfect conductor. The opposite is to assume that the earth is a perfect insulator. Then there is no exchange of heat between the various lattitudes, which receive quite different amounts of radiation. Let $\phi(\theta)$ be the angle between the direction of the sun and the normal of the surface at a lattitude θ. Then we introduce the function $f(\theta)$ as

$$f(\theta) = 4 \langle \cos(\phi(\theta)) \rangle, \tag{9}$$

where the average is over the duration of the day and the days of the year. The 4 is for convenience since we know that the average over all lattitudes equals 1/4. Thus $f(\theta)$ gives the fraction of the radiation that enters at lattitude θ. Then we have the thermal out-flux $F_{\mathrm{out}}(\theta)$ equal to solar influx at lattitude θ

$$F_{\mathrm{out}}(\theta) = S_{\mathrm{in}}\, f(\theta). \tag{10}$$

with S_{in} given by Eq. (7). The same amount has to leave at the top of the atmosphere. The atmosphere has at all lattitudes the same composition. So we may use the same ratio R for all lattitudes. Thus the surface temperature at lattitude θ equals

$$T(\theta) = R\, T_{\mathrm{out}}\, f(\theta)^{1/4} = 288\, f(\theta)^{1/4}\mathrm{K}. \tag{11}$$

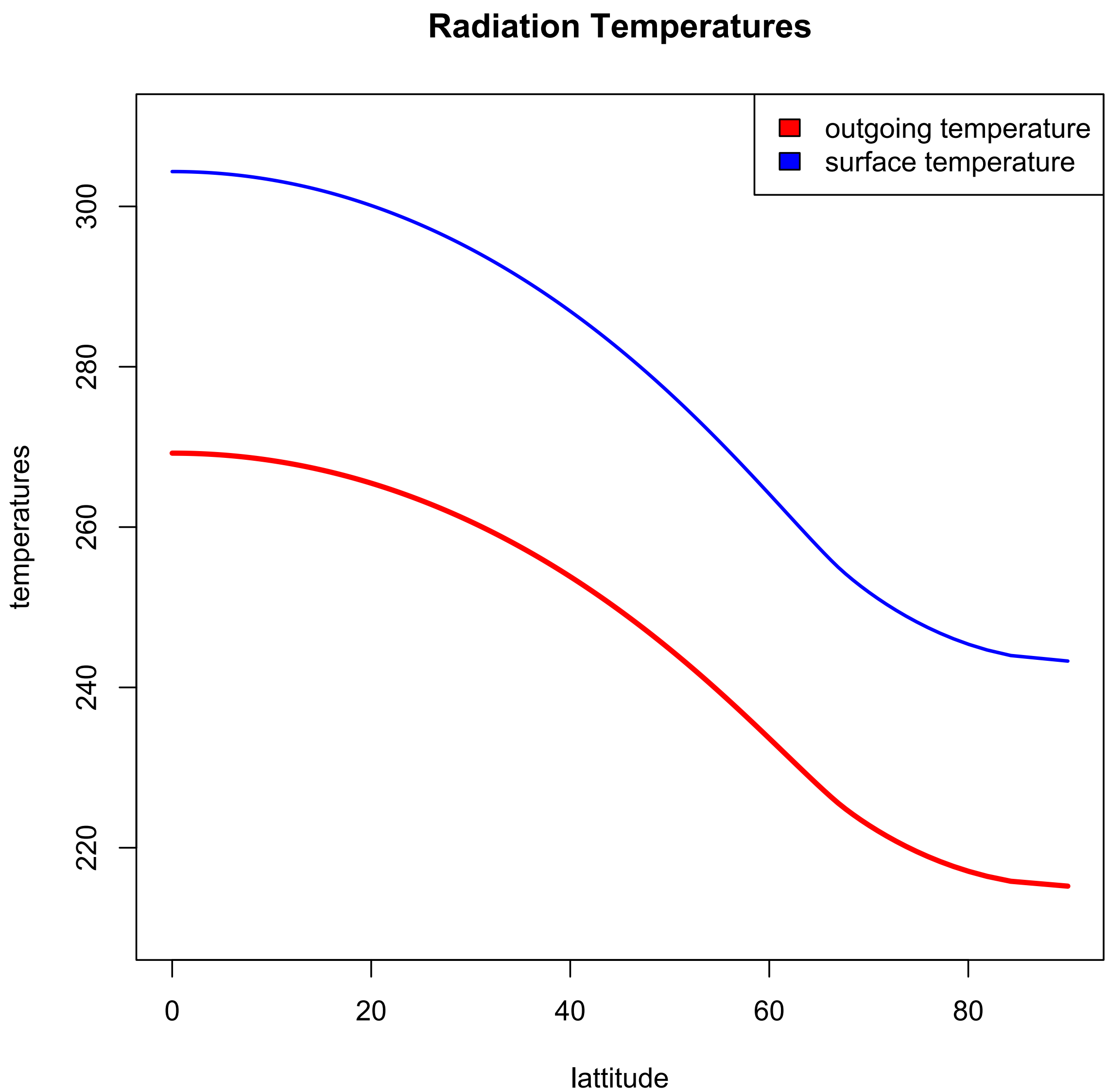

Figure 2: Yearly average of the radiation as function of the lattitude θ

The curve $f(\theta)^{1/4}$ is also shown in Fig. 2. Although the variation in $f(\theta)$ is substantial, the 1/4 power makes the deviations from 1 much smaller. Translating them with Eq. (11) into temperatures we find 31 ^{o}C at the equator and -30 ^{o}C at the poles. In reality the differences are smaller as can be expected since heat flows from the equator to the poles leveling the temperature differences. But this temperature profile is closer to the reality than the constant value of Arrhenius.

3 The Primary Stream

The surface of the earth emits thermal radiation in response to the influx of solar radiation in the visible frequency range. This radiation $I_0(z,\theta,\nu)$ is a function of the height z, the angle θ with the normal to the earth and the frequency ν. At the groundlevel $z=0$ it is independent of θ and the frequency distribution is given by the (normalized) Planck distribution

$$P_{\rm Pl}(\nu) = \frac{15}{\pi^4}\left(\frac{h}{k_B T}\right)^4 \frac{\nu^3}{\exp(h\nu/k_B T)-1} \tag{12}$$

Thus at $z=0$ it has the shape

$$I_0(\nu,\theta;0) = I_0\,\frac{1}{2\pi}P_{\rm Pl}(\nu) \tag{13}$$

On its way to higher layers it is partially absorbed by the greenhouse gasses with absorption cross-section $\sigma(\nu)$. Multiplying the cross-section by the density $n(z)$ we get the inverse of the absorption length $l_{\rm abs}(z,\nu)$

$$l_{\rm abs}(z,\nu) = \frac{1}{n(z)\sigma(\nu)} \tag{14}$$

The absorption causes the radiation to decay in an exponential way

$$I_0(\nu,\theta;z) = G_\uparrow(\nu,\theta;z,0)I_0(\nu,\theta;0), \tag{15}$$

where the propagator G is given by

$$G_\uparrow(\nu,\theta;z,z') = \exp\left(-\int_{z'}^{z}\frac{dz''}{\cos(\theta)}n(z'')\sigma(\nu)\right). \tag{16}$$

The $\cos(\theta)$ in the denominator accounts for the longer path of a direction θ. The density is an exponential function of the height

$$n(z) = n(0)\exp(-z/l_{\rm atm}), \tag{17}$$

where $l_{\rm atm}$ is the "height" of the atmosphere (11 km). So the integration in the exponent of G can be carried out

$$G_\uparrow(\nu,\theta;z,z') = \exp\left(-\frac{q(\nu)}{\cos(\theta)}[\exp(-z'/l_{\rm atm}) - \exp(-z/l_{\rm atm})]\right). \tag{18}$$

Here $q(\nu)$ is the dimensionless absorption strength

$$q(\nu) = l_{\rm atm}/l_{\rm abs}(0,\nu) = l_{\rm atm}n(0)\sigma(\nu). \tag{19}$$

4 The Secundary Streams

The layer (z, dz) is absorbing and emitting. The absorption is proportional to the strength $I_{\rm in}(\nu, \theta; z)$ of the radiation at level z multiplied by the absorption coefficient $n(z)\sigma(\nu)$. The emitted radiation has an amplitude[1] $J(z)$, which is independent of the direction θ and thermally distributed over the frequencies. Conservation of energy then requires that, integrated over frequencies and angles

$$J(z)dz = n(z)dz \int d\nu\sigma(\nu)\, 2\pi \int \sin\theta d\theta\, I_{\rm in}(\nu, \theta; z). \tag{20}$$

On the right hand side we have the amount of energy absorbed in the layer (z, dz) and on the left hand side the amount of energy that is emitted in layer (z, dz).

The incoming flux $I_{\rm in}$ is composed of three currents

$$I_{\rm in}(\nu, \theta; z) = I_0(\nu, \theta; z) + I_\uparrow(\nu, \theta; z) + I_\downarrow(\nu, \theta; z). \tag{21}$$

The first is the primary current coming from the surface of the earth defined in Eq. 15. $I_\uparrow$ is the secundary upstream coming from the layers below z and $I_\downarrow$ is the downstream coming from the layers above z.

The upstream $I_\uparrow$ is the inregral of the emitted radiation at level $z' < z$ in the upward direction θ

$$I_\uparrow(\nu, \theta; z) = \int_0^z dz' G_\uparrow(\nu, \theta; z, z') \frac{J(z')}{4\pi} P_{\rm Pl}(\nu). \tag{22}$$

The strength of the emitted radiation at level z' equals $J(z')$, which is lmdependent of the direction θ and which has a thermal distribution over the frequencies ν. Likewise the downstream

$$I_\downarrow(\nu, \theta; z) = \int_z^\infty dz' G_\downarrow(\nu, \theta; z, z') \frac{J(z')}{4\pi} P_{\rm Pl}(\nu), \tag{23}$$

where the down propagator is given by

$$G_\downarrow(\nu, \theta; z, z') = \exp\left(-\frac{q(\nu)}{|\cos(\theta)|}[\exp(-z/l_{\rm atm}) - \exp(-z'/l_{\rm atm})]\right). \tag{24}$$

J is expressed by Eq. (20) in terms of $I_{\rm in}$, which is the sum of the three currents $I_0, I_\uparrow$ and $I_\downarrow$, which in turn depend by Eq. (22) and (23) on J. So Eq. (20) is an integral equation for J. It reads

$$J(z) = n(z) \int d\nu\sigma(\nu)\, 2\pi \int \sin\theta d\theta\, \left[I_0(\nu, \theta; z) + I_\uparrow(\nu, \theta; z) + I_\downarrow(\nu, \theta; z)\right]. \tag{25}$$

The study of this integral relation is the core of this note.

We have have two important combination of the streams. The first is the total radiation intensity $I(z)$[2]

$$I(z) = I_0(z) + I_\uparrow(z) + I_\downarrow(z). \tag{26}$$

This intensity will be related to the temperature of the layer z. The second stream is the upward flux $F(z)$

$$F(z) = I_0(z) + I_\uparrow(z) - I_\downarrow(z) = F_{\rm out}. \tag{27}$$

[1] The dimension of J is flux/length

[2] Rather than introducing new symbols for integrated quantities, we adopt the convention to use the same symbol but give it less arguments. So $I(\nu; z)$ is $I(\nu, \theta; z)$ integrated over the angles and $I(z)$ equals $I(\nu; z)$ integrated over the frequencies.

Conservation of energy requires this flux to be constant. Here $F_{\rm out}$ is the local influx of the sun. On the average over the earth it equals $S_{\rm in}$ given in Eq. (7). As the ingredients of the flux $F(z)$ are linearly depdendent on $J(z)$, Eq. (27) is also an integral equation for $J(z)$. So the relations (20) and (27) are intimately related, as both express the conservation of energy.

5 The integral over angles

We can simplify the structure of the integral equation a bit, using the fact that θ only occurs in the propagators. So we encounter the integral

$$G_\uparrow(\nu; z, z') = \int_0^{\pi/2} \sin\theta \, d\theta \, G_\uparrow(\nu, \theta; z, z'). \tag{28}$$

Using $u = \cos\theta$ as integration variable we can relate the integral to the exponential integrals

$$E_n(x) = \int_1^\infty \frac{dy}{y^n} \exp(-xy) \tag{29}$$

Since

$$G_\uparrow(\nu; z, z') = \int_0^1 du \exp\left(-\frac{q(\nu)}{u}[\exp(-z'/l_{\rm atm}) - \exp(-z/l_{\rm atm})]\right). \tag{30}$$

the substitution $u = 1/s$ gives

$$G_\uparrow(\nu; z, z') = E_2\left(-q(\nu)[\exp(-z'/l_{\rm atm}) - \exp(-z/l_{\rm atm})]\right). \tag{31}$$

These averaged propagators yields the averaged currents

$$\left\{ \begin{array}{rcl} I_0(\nu; z) & = & G_\uparrow(\nu; z, 0) I_0 P_{\rm Pl}(\nu), \\ I_\uparrow(\nu; z) & = & \frac{1}{2}\int_0^z dz' G_\uparrow(\nu; z, z') J(z') P_{\rm Pl}(\nu), \\ I_\downarrow(\nu; z) & = & \frac{1}{2}\int_z^\infty dz' G_\downarrow(\nu; z, z') J(z') P_{\rm Pl}(\nu). \end{array} \right. \tag{32}$$

Using these averaged flows the integral equation for $J(z)$ becomes

$$J(z) = n(z) \int d\nu \sigma(\nu) \, \left(I_0(\nu; z) + I_\uparrow(\nu; z) + I_\downarrow(\nu; z)\right), \tag{33}$$

with the averaged currents given by Eq. (32). A general method of solution is by iteration, with $I_0(\nu; z)$, which is independent of J, as start. Feeding that into the expressions for $I_\uparrow(\nu; z)$ and $I \downarrow (\nu; z)$ gives the next approximation. Iteration mimicks the physical process which starts with the primary radiation stream from the surface of the earth, that excites the secundary streams.

6 Radiation Balance in a gray Atmosphere

The simplest representation of the absorption in the atmosphere is the so-called gray atmosphere, which has an uniform cross-section $\sigma(\nu)$ over the whole frequency range.

Then $q(\nu)$ is a constant value q and the average over ν becomes a trivial integration over the Planck distribution. So the integral equation for $J(z)$ gets the form

$$J(z) = \frac{q\exp(-z/l_{\rm atm})}{l_{\rm atm}} \left(I_0 G_\uparrow(z,0) + \frac{1}{2}\int_0^z dz' G_\uparrow(z,z')J(z') + \frac{1}{2}\int_z^\infty dz' G_\downarrow(z,z')J(z') \right), \tag{34}$$

with the simplified propagators

$$\left\{ \begin{array}{rcl} G_\uparrow(z,z') & = & E_2(q[\exp(-z'/l_{\rm atm}) - \exp(-z/l_{\rm atm})]), \\ G_\downarrow(z,z') & = & E_2(q[\exp(-z/l_{\rm atm}) - \exp(-z'/l_{\rm atm})]). \end{array} \right. \tag{35}$$

One now can take advantage from using the optical thickness τ as height variable instead of z

$$\tau(z) = q\exp(-z/l_{\rm atm}), \qquad z(\tau) = -l_{\rm atm}\log(\tau/q), \qquad dz = -l_{\rm atm}\frac{d\tau}{\tau}. \tag{36}$$

In terms of the variable τ the integral equation becomes

$$j(\tau) = I_0 E_2(q-\tau) + \frac{1}{2}\int_\tau^q d\tau' E_2(\tau'-\tau)j(\tau') + \frac{1}{2}\int_0^\tau d\tau' E_2(\tau-\tau')j(\tau'), \tag{37}$$

where $j(\tau)$ stands for the combination

$$j(\tau) = l_{\rm atm} J(z(\tau))/\tau. \tag{38}$$

Again this equation may be solved by iteration with $I_0 E_2(\tau - q)$ as start.

In order to get an insight in the solution we approximate E_2

$$E_2(x) = \int_1^\infty \frac{dy}{y^2}\exp(-xy) = \exp(-x) - xE_1(x), \tag{39}$$

by the first term. As this approximation is only for mathematical tractability we comment one its significance later. Then the equation is exactly solvable. Aplying the operator

$$\mathcal{O}(\tau) = e^{-\tau}\frac{d}{d\tau}e^{2\tau}\frac{d}{d\tau}e^{-\tau}, \tag{40}$$

to both sides of the equation gives

$$j(\tau) - \frac{d^2 j(\tau)}{d\tau^2} = j(\tau), \tag{41}$$

with the possible solutions

$$j(\tau) = a\tau + b. \tag{42}$$

The constants a and b are determined by inserting the solution given by Eq. (42) into the original equation Eq. (37). This gives two equations

$$a = b, \qquad I_0 = \frac{1}{2}a(1+q) + \frac{1}{2}b = a(1+q/2). \tag{43}$$

So we have still one undetermined parameter for which we can take a or I_0. This parameter follows on the conditions on the flow in de vertical direction.

7 The Energy Flow in the vertical direction

The energy flow in the vertical direction is the sum of the averaged primary current $I_0(z)$ and the secundary upstream $I_\uparrow(z)$ minus the secondary dowmstream $I_\downarrow(z)$. The net flow upwards is given by

$$F(z) = I_0(z) + I_\uparrow(z) - I_\downarrow(z) = F_{\text{out}}, \tag{44}$$

where F_{out} is the (local) influx from the sun, i.e. 243 W/m^2 for the earth on the average (see Eq.(7)).

As the three currents are related to $J(z)$ by Eq. (32), or equivalently to $j(\tau)$. For the case of the gray atmsosphere with the simplified propagator we find

$$\begin{cases} I_0(z) &= I_0 G_\uparrow(z,0) &= I_0 \exp(\tau - q), \\ I_\uparrow(z) &= \dfrac{1}{2}\displaystyle\int_\tau^q \exp(\tau' - \tau) j(\tau) &= \dfrac{a}{2}[(2+\tau) - (2+q)\exp(\tau - q)], \\ I_\downarrow(z) &= \dfrac{1}{2}\displaystyle\int_0^\tau \exp(\tau - \tau') j(\tau) &= \dfrac{a}{2}\tau. \end{cases} \tag{45}$$

Inserting the values of I_0 given by Eq. (43) we observe that $F(z)$ is indeed a constant and that a is given by

$$a = F_{\text{out}}. \tag{46}$$

This settles the values of the constants, for instance

$$I_0 = F_{\text{out}}(1 + q/2). \tag{47}$$

The amplitude of the primary current from the surface of the earth is the sum of the solar influx and the downstream generated by the absorption in the atmosphere.

8 Local Thermodynamic Equilibrium

Sofar the discussion concerns only the radiation, but for the greenhouse effect the temperature is important. The connection between radiation and temperature is established through the requirement of local thermodynamic equilibrium. In our case it means that the total strength of the radiation $I(z)$

$$I(z) = \int d\nu \int \sin\theta d\theta I_{\text{in}}(\nu, \theta z), \tag{48}$$

corresponds to the local temperature $T(z)$ according to the law of Stefan-Boltzmann

$$I(z) = \sigma_{\text{SB}} T^4(z). \tag{49}$$

Since $I_\downarrow(\infty) = 0$ the value of $I(\infty) = F(\infty)$. So we can associate with the outflow at the top of the atmosphere a temperature

$$T(\infty) = \left(\frac{F(\infty)}{\sigma_{\text{SB}}}\right)^{1/4} \tag{50}$$

The greenhouse effect is best represented by the ratio of the temperature at the surface and that at of the top of the atmosphere.

$$R = \frac{T(0)}{T(\infty)} = \left(\frac{I(0)}{I(\infty)}\right)^{1/4}. \tag{51}$$

For the grey atmosphere we find with Eq. (46)

$$R = (1+q)^{1/4} \tag{52}$$

For the present $R = 1.125$ we would need $q = 0.6$.

9 The semi-gray atmosphere

The gray atmosphere implies a single absorption cross-section σ for all frequencies. In reality the absorption of the radiation is strongly frequency dependent and concentrated in absorption lines. So, for the gray atmosphere the σ must be seen as the average absorption in the thermal frequency band. A step in the direction of a more realistic model is to consider two windows in the spectrum: one with a (uniform) absorption coefficient σ and the other window with frequencies that are not absorbed by the atmosphere.

With the introduction of two regions of different absorption a new parameter r comes into the problem. It is the size of the absorbing part of the spectrum. We define it as

$$r = \int_{\rm abs} d\nu P_{\rm Pl}(\nu), \tag{53}$$

where the integration is over the absorbing frequencies. Each flow is decomposed into two parts, distinguished by a superscript a for the absorbing frequencies and o for the frequencies that that propagate unhindered through the atmosphere. E.g.

$$I_0(z) = I_0^o(z) + I_0^a(z) = (1-r)I_0 + rI_0 G_\uparrow(z,0). \tag{54}$$

Instead of the integral equation (33) we get

$$J(z) = rn(z)\sigma \left(I_0^a(z) + I_\uparrow^a(z) + I_\downarrow^a(z) \right), \tag{55}$$

There are two important differences with respect to Eq. (33): the factor r in front due to the integration over ν and the appearance of the partial currents I^a instead of the full currents.

We now can follow the same path as for the gray atmosphere. So make the transition from the height z to the optical depth τ as in Eq. (36). The resulting equation for $j(\tau)$ then reads

$$j(\tau) = rI_0 \exp(q-\tau) + \frac{r}{2}\int_\tau^q d\tau' \exp(\tau'-\tau) j(\tau') + \frac{r}{2}\int_0^\tau d\tau' \exp(\tau-\tau') j(\tau'). \tag{56}$$

Then apply the same operator Eq. (42) to this equation with the result

$$j(\tau) - \frac{d^2 j(\tau)}{d\tau^2} = rj(\tau). \tag{57}$$

The solution of this equation is the sum of two exponentials

$$j(\tau) = A_+ \exp(\tau\alpha) + A_- \exp(-\tau\alpha), \tag{58}$$

with

$$\alpha = \sqrt{1-r}. \tag{59}$$

The two constants $A_\pm$ can be linked to I_o, as in the case of the gray atmosphere, by inserting the possible solution Eq. (58) into the Eq. (56). It is more direct to require that the solution Eq. (58) fulfils the flux equation (44) or

$$F_{\text{out}} = I_0^o(z) + I_0^a(z) + I_\uparrow^o(z) + I_\uparrow^a(z) - I_\downarrow^o - I_\downarrow^a. \tag{60}$$

The currents are given by Eq. (54) and

$$\begin{cases} I_\uparrow^o(z) &= \dfrac{\alpha^2}{2}\displaystyle\int_\tau^q d\tau' j(\tau') \\ I_\uparrow^a(z) &= \dfrac{1-\alpha^2}{2}\displaystyle\int_\tau^q d\tau' \exp(\tau-\tau') j(\tau'), \\ I_\downarrow^o(z) &= \dfrac{\alpha^2}{2}\displaystyle\int_0^\tau d\tau' j(\tau'), \\ I_\downarrow^a(z) &= \dfrac{1-\alpha^2}{2}\displaystyle\int_0^\tau d\tau' \exp(\tau'-\tau) j(\tau'). \end{cases} \tag{61}$$

We have eliminated r in favor of the more convenient α. Inserting the solution (57) into the flux equation leads to exponential terms of the type $\exp(\pm\alpha\tau), \exp(\tau-q), \exp(-\tau)$ and constant terms.

$$\begin{cases} I_\uparrow^o(z) &= A_+\dfrac{\alpha}{2}[e^{\alpha q} - e^{\alpha\tau}] + A_-\dfrac{\alpha}{2}[e^{-\alpha\tau} - e^{-\alpha q}] \\ I_\uparrow^a(z) &= A_+\dfrac{1+\alpha}{2}[e^{\alpha\tau} - e^{\tau-q+\alpha q}] + A_-\dfrac{1-\alpha}{2}[e^{-\alpha\tau} - e^{\tau-q-\alpha q}], \\ I_\downarrow^o(z) &= A_+\dfrac{\alpha}{2}[e^{\alpha\tau} - 1] + A_-\dfrac{\alpha}{2}[1 - e^{-\alpha\tau}] \\ I_\downarrow^a(z) &= A_+\dfrac{1-\alpha}{2}[e^{\alpha\tau} - e^{-\tau}] + A_-\dfrac{1+\alpha}{2}[e^{-\alpha\tau} - e^{-\tau}]. \end{cases} \tag{62}$$

Those with $\exp(\pm\alpha\tau)$ cancel. The three other types lead to three conditions for I_0 and the $A_\pm$. The terms with $\exp(-\tau)$ give the condition

$$A_+(1-\alpha) + A_-(1+\alpha) = 0. \tag{63}$$

The terms with $\exp(\tau - q)$ give the relation

$$I_0 = A_+\frac{\exp(\alpha q)}{2(1-\alpha)} + A_-\frac{\exp(-\alpha q)}{2(1+\alpha)} \tag{64}$$

and the constant terms lead to the condition

$$\alpha^2 I_0 + A_+\frac{\alpha}{2}[\exp(\alpha q) + 1] - A_-\frac{\alpha}{2}[\exp(-\alpha q) + 1] = F_{\text{out}}. \tag{65}$$

The solution of the three equations (61),(62) and (63) reads

$$A_+ = \frac{2(1+\alpha)}{\alpha}\det(q,\alpha), \qquad A_- = -\frac{2(1-\alpha)}{\alpha}\det(q,\alpha). \tag{66}$$

with the expression for $\det(q,\alpha)$

$$\det(q,\alpha) = \frac{1+\alpha}{1-\alpha}\exp(q\alpha) + \frac{1-\alpha}{1+\alpha}\exp(-q\alpha) + 2. \tag{67}$$

With the $A_\pm$ known as function of α (or r) and q all the currents are known.

10 The Greenhouse Effect

For the greenhouse effect we need the total radiation $I(z)$

$$I(z) = I_0(z) + I_\uparrow(z) + I_\downarrow(z), \tag{68}$$

which is linked to the temperature by Eq. (49). In particular we need the values at $z = 0$ and $z = \infty$ for the greenhouse ratio R

$$R = \frac{T(0)}{T(\infty)} = \left(\frac{I(0)}{I(\infty)}\right)^{1/4} = \left(\frac{I_0(0) + I_\downarrow(0)}{I_0(\infty) + I_\uparrow(\infty)}\right)^{1/4} = \left(\frac{F_{\text{out}} + 2I_\downarrow(0)}{F_{\text{out}}}\right)^{1/4}. \tag{69}$$

In Fig. 3 we show R as function of q and r. Note that we recover for $r \to 1$ or $\alpha \to 0$ the result of the gray atmosphere $R \to (1+\tau)^{1/4}$. For smaller values of r the value of R saturates for increasing q.

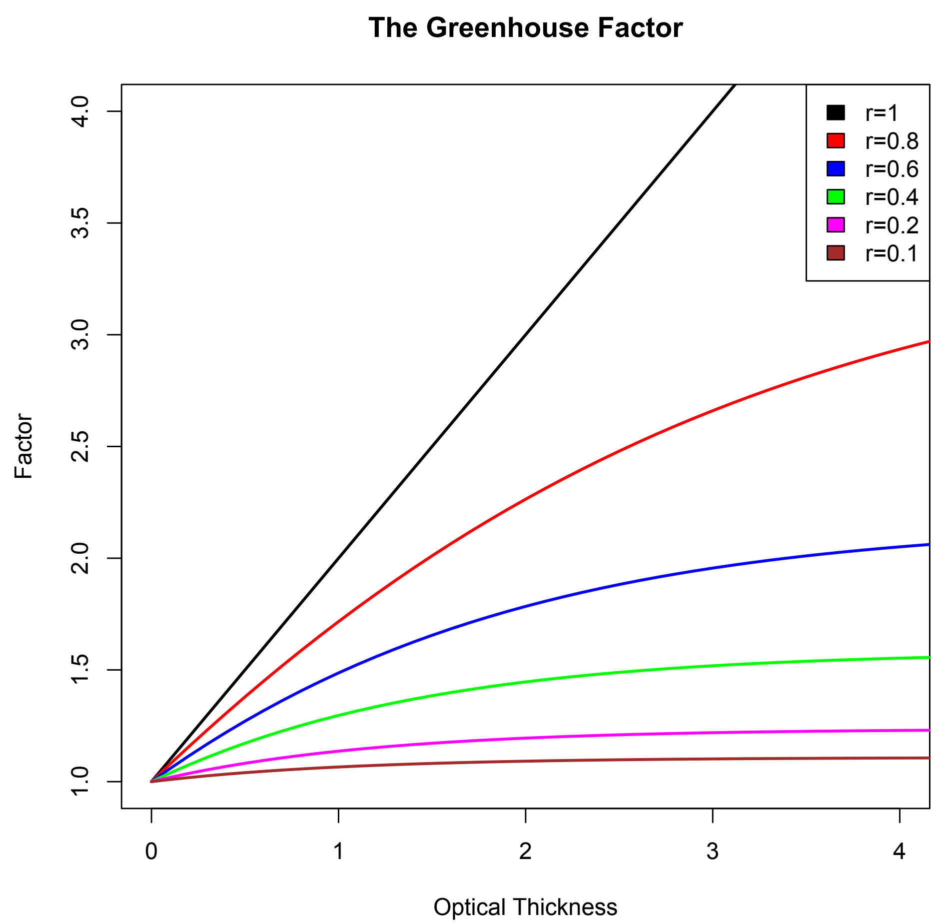

Figure 3: The Greenhouse Factor $R(q, r)$ as function of the absorption coefficient q (optical thickness) and the absorption range r.

Another signature of the greenhouse effect is the change in the spectrum of the outgoing radiation. Without absorption the spectrum would be distributed according to the Planck distribution, with the temperature based on the strength of F_{out}. Due to the absorption the absorbing frequences are depleted in favor of the transparant frequencies. The depleted frequences are represented by

$$I_0^a(\infty) + I_\uparrow^a(\infty) = \frac{4}{\det(q, \alpha)} F_{\rm out} \tag{70}$$

The transparant frequencies that benefit of the absorption are given by

$$I_0^o(\infty) + I_\uparrow^o(\infty) = \left(1 - \frac{4}{\det(q, \alpha)}\right) F_{\rm out} \tag{71}$$

Note that the sum of the currents on the left hand side of Eq. (70) and (71) equals $F_{\rm out}$ as it should on the basis of the flow equation. Since $\det(q, \alpha)$ grows with the absorption q the part of the absorbing spectrum gradually decreases in favor of the transparant part.

We get the intensities of the radiation in the two regimes by dividing them by their size: $r = 1 - \alpha^2$ for the absorbing part and $1 - r = \alpha^2$ for the transparant part. So the spectrum reads for the transparant part

$$I^o(\nu; \infty) = \frac{1}{\alpha^2} \left(1 - \frac{4}{\det(q, \alpha)}\right) P_{\rm Pl}(\nu), \tag{72}$$

and for the absorbing part

$$I^a(\nu; \infty) = \frac{4}{(1 - \alpha^2) \det(q, \alpha)} P_{\rm Pl}(\nu), \tag{73}$$

In Fig. 4 we have shown the outgoing spectrum for a number of absorptions q and a value of $r = 0.11$. The dent in the spectrum has sharp walls since we have a sharp distinction between absorbing and transparant frequencies. One observes that for $q > 4$ the absorbing frequencies are almost completely removed from the spectrum (saturation), while $q = 4$ is still a weak absorption with an absorption length of 3 km.

11 Conclusion

We have studied the radiative balance of a semi-gray atmosphere, consisting out of a region with a constant absorption coeffcient and a background of transparant frequencies. The absorbing region may exist out of a number of separate absorbing lines. For the exact solution it is necessary that the absorbing lines are of equal strength. Such an atmosphere is characterized by two parameters: the strength q of the absorption and the size r of the absorbing region.

The greenhouse effect is represented as a ratio R of the temperature at the surface of the earth and at the top of the atmosphere, where the temperature is determined by the radiative outflow. R is a more convenient parameter than the usual temperature difference between the bottom and the top of the atmosphere, which is e.g. lattitude dependent, while R only depends on the composition of the atmosphere. The dependence of R on r and q is shown in Fig. 3. The general tendency is that for fixed and small r the ratio R increases with q but soon saturates. For $r = 1$ (gray atmosphere) it keeps rising with q.

The spectrum of the outgoing radiation gives a more detailed picture of the effect of the absorption. The radiation of the absorbing frequencies is reduced in favor of the radiation of the transparant frequencies. This is clearly demonstrated in Fig. 4 where we show the effect of an absorbing region between the frequencies 620 cm^{-1} and 720 cm^{-1}.

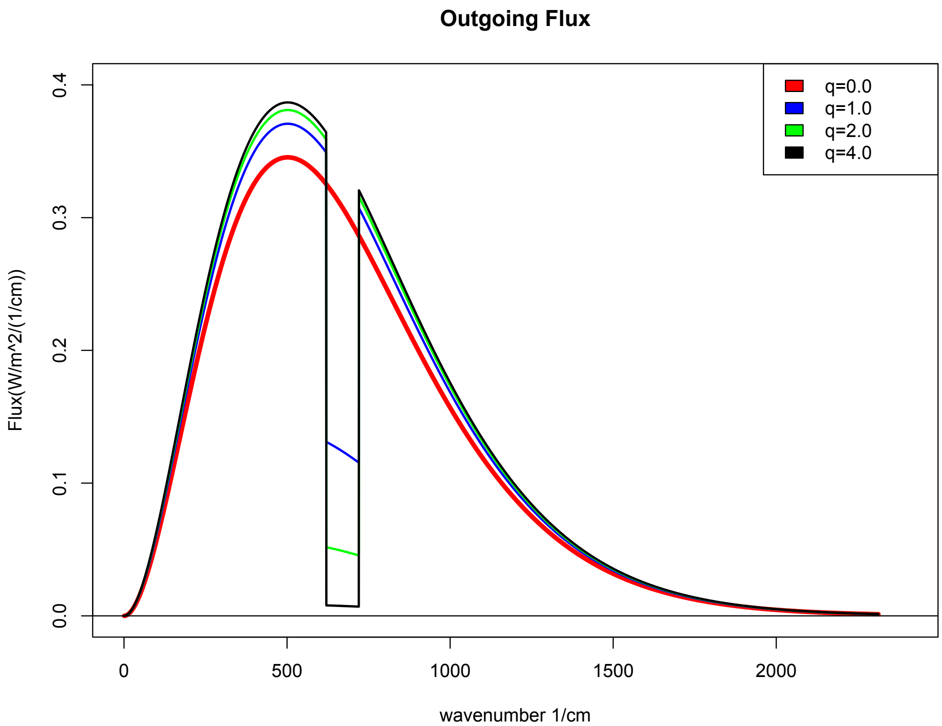

Figure 4: The spectrum of the outgoing radiation for a number of absorptions q and a value of $r = 0.11$

That region corrresponds to the absorption band of symmetric bending mode of CO_2. Again one observes that the effect of more CO_2 saturates above $q = 4$, which is still a rather weak absorption with an absorption length of some 3 km. A higher concentration of CO_2 has influence on the width of the absorbing region. According to Arrhenius the width increases logarithmically with the concentration.

We have made a mathematically simplification by replacing $E_2(x)$ in Eq. (41) by the first term. This is equivalent by replacing the angular integral by the sum over the vertical directions. As the absorption in the vertical direction is less than in an oblique direction we have made the atmosphere more transparant. Keeping $E_2(x)$ will qualitative give the same results, but requires numerical solutions of the integral equations.

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